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ABSTRACT

The purpose of this project is to teach learning and understanding of mathematics at grades seven through nine through the use of science experiments. Previous knowledge of science on the part of students or teachers is not necessary. Lists of needed equipment are found at the beginning of this volume. It is strongly recommended that the teacher try out each experiment before it is done in class. The experiments in part one involve basic measurements of length, mass, time, and temperature. The material can be covered in three or four weeks. Included in the Teacher's Commentary are background information, discussion of activities and exercises, and answers to problems. (RH)

SCHOOL
MATHEMATICS
STUDY GROUP

MATHEMATICS THROUGH SCIENCE

PART I: MEASUREMENT AND GRAPHING

TEACHERS' COMMENTARY

(revised edition)

SMSG

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Part I: Measurement and Graphing

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FOREWARD

During the summer of 1963, a group of fifteen mathematicians, scientists, and teachers, working under the auspices of the School Mathematics Study Group, prepared experimental textbook units which explored the possibility of developing some of the basic concepts of mathematics through simple, but significant, physical science experiments. These units were tested in representative classrooms in a number of centers over the country during the following school year. In the summer of 1964, revisions were made on the basis of the results of these trials.

The purpose of this project was to see if the learning and understanding of mathematics at the grade levels seven through nine could be improved through the use of the approach through science experiments. The results of the preliminary tests were quite encouraging. The students found the experiments fascingting and learned the related mathematical principles quickly and easily. Many suggestions for changes were made by teachers on the basis of the reactions of their students. These suggestions were incorporated into this revision wherever possible.

It should be noted that these units are to be used in the mathematics classroom and that they are primarily designed to teach mathematical concepts rather than those of science. It is true that the procedures and principles of science included are sound and correct within the framework in which they are used. The experience in science which the student will gain from the study of these units will no doubt be useful in subsequent courses in the physical sciences, but the main purposes of the units are to teach mathematics.

Previous knowledge of science on the part of students or teachers is not necessary, although any knowledge which they might have in this field will be useful. The experimental procedures are clearly described. Lists of needed equipment will be found at the beginning of this volume. Many substitutions are possible in these lists since the experimental equipment is not at all critical. The teacher is warned, however, that there is a natural perversity in experimental work. It is strongly recommended that the teacher try out each experiment before it is done by the class, particularly if equipment substitutions must be made.

The experiments in this volume involve basic measurements of length, mass, time and temperature. Though most of the experiments will be described in terms of the metric system of measurement, British units are not ignored. It is assumed that these experiments can be done in the mathematics classroom. Some will be done by all of the students working in small groups, and some will have to be demonstrations done by the teacher.

The use of the data collected is an important aspect of each experiment. These data will be measurements which can be graphed on a single number line or on a set of rectangular coordinates. This leads, in a completely natural way, to the mathematical concepts of approximation, units of measurement, inequalities, functions, and graphing. The fitting of a line to the graph of the experimental data leads to the general concept of lines and the slope and equation of lines.

The material in this textbook can be covered in a three or four week period. The teacher can best decide at what time of the course its use might be most appropriate. These units supplement, but do not replace, whatever mathematics book is in regular use in the classroom. It may, however, be possible to omit some sections of the regular textbook if the teacher feels that the treatment of the topics in this text is sufficient.

TABLE OF CONTENTS

EQUIPMENT LIST

Part I

Sources for equipment in the following experiments are indicated below and are coded at the right of each item.

(1) Scientific supply (i.e., Cenco or Welch, etc.)

(2) Hardware store

(3) Stationery store

(4) Variety store

(5) Home

Chapter 1

INTRODUCTION TO MEASUREMENT

1. Properties of Order Experiment - Teacher Demonstration

1 meter stick (1) or yard stick (2)

1 triangular file (5 in.) extra slim taper - (2) or 1 knitting needle

2 #25 binder clamps - (3)

1 paper clip, spring-type (Hunt Brand #1) (3)

1 box #1 "Gem" paper clips (3)

2 rubber bands (3)

4 "Dixie" cups (6 oz.) (4)

6 $\frac{3}{8}$ in wooden dowels (3' lengths) (2)

Chapter 2

LENGTH AND THE NUMBER LINE

1. The Unmarked Stick Experiment - Students are to work individually. Each student should have the following equipment.

1 straight stick (approximately 15") (5)

2 sheets of notebook paper (lined) (3)

1 sheet of graph paper (10 squares per inch) (3)

1 ball of string (4)
1 small roll of masking tape (4) } for entire class

Chapter 3

RELATIONS, FUNCTIONS AND GRAPHING

1. Cantilever Experiment - Students are to work in groups of two. Each group should have the following equipment.

5 books, identical (with dust jackets, if possible)
or 5 wooden boards (1" × 8" × 10")
1 foot ruler, with a metric scale (4)

2. Irregular Bottle Experiment - Same instructions as 1.

1 irregular-shaped bottle (12 fluid oz) (5)
1 plastic pill bottle (approximately 20 cm^3) (5)
1 "Magic Mending" tape, Scotch brand ($\frac{3}{4}$" × 1296") (4)
1 foot ruler, with a metric scale (4)
1 #10 can or 1 gallon water container

Chapter 4

LINEAR FUNCTIONS

1. The Coat Hanger Experiment - Students are to work in groups of four. Each group should have the following equipment.

1 hook weight (100 gram) (1) Cenco No. 9810
2 hook weights (200 gram) (1)
1 hook weight (500 gram) (1)
1 coat hanger (wire) (5)
2 paper clips, jumbo "Gem" (3)
1 foot ruler, with metric scale (4)

2. The Loaded Spring - Same instructions as 1.

1 hook weight (100 gram) (1)
2 hook weights (200 gram) (1)
1 hook weight (500 gram) (1)
1 spring (2)
1 pencil (unsharpened)
1 roll Scotch tape (4)
1 foot ruler, with metric scale (4)
2 sheets of note paper (3) or (4)

3. Centigrade-Fahrenheit - Same instructions as 1.

1 thermometer, Centigrade (-20° to 110°) (1) Cenco No. 19240 - 1

1 thermometer, Fahrenheit (0° to 230°) (1) Cenco No. 19282 - 1

1 #10 tin can (for liquids)

ice

salt

Chapter 1

MEASUREMENT

1.1 Introduction

Chapter I is intended to fix in the student's mind the more obvious facts of measurement which he has gained from his life experience thus far and to develop new mathematics from them. At the same time questions are raised which are intended to lead the student to a better understanding of the philosophy of measurement and to a development of the related mathematics.

1.2 Measurements, Magnitudes and Units

In this chapter the student will probably encounter his first serious study of measurements. Although he is sure to have used such phrases as "two miles", "17 seconds", and "30 acres" very often, it is unlikely that he will recognize much of the mathematical nature of such concepts.

It is possible, however, that some of your students will have confused the idea of number and measurement in their own minds and they cannot see any difference between the two. If so, you must first clarify these ideas. It will be necessary to explain resemblances between number and measurement if your students think of them as identical.

Perhaps the most helpful thing you can do to make the necessary distinction is to emphasize the different uses to which numbers and measurements are put. The students may not really know what numbers are, although they can use them well for many purposes. The same can be said for measurements. Challenge the student who confuses the two to describe the distance to his home, the length of the class period, or the area of the floor in the classroom, by means of a number alone. His inability to do so should convince him that measurements are needed to convey some kinds of information that numbers cannot do. Conversely, measurements are inappropriate where only numbers can serve.

The distinction between number and measurement can be made a bit sharper by turning attention to procedures of measurement. The students already know something about these procedures. The term measurement can be given crude intuitive meaning as "the result of a measuring process". It may be that the students already use some other term such as quantity or amount to mean the same thing. It is useful to point out that measurement cannot be done by pencil and paper alone. It must involve experimentation with actual things or events. More specifically, the experiment is an act of comparison of two objects or events requiring a careful set of instructions for its performance. It is unlikely that your students will appreciate how detailed those instructions must be. Sections 1.10, 1.11 and 1.12 will lead them through the necessary steps for length measurement. Other measurements are determined by experiments with still different sets of instructions. A strong general impression should be left with the students that measurement cannot be done strictly within the framework of mathematics but involves science as well.

It is conceivable that some students will have thought deeply enough at this stage to ask the following question: "If lengths are not numbers, just exactly what are they?" This is the type of question the authors of the text have dealt with and have attempted to answer with the help of the students and their teachers. It is hoped that the student will be able to arrive at his own understanding of measurement at the conclusion of this course.

For the teacher, however, knowledge at a deeper level is desirable. The following explanation outlines a suitable answer to the question, "What is a length?"

A length of seven feet, we must agree at first, is not a particular physical object. It cannot be seen or put on display. We can perceive with our senses a seven-foot bed, a seven-foot oar, or a seven-foot fishing rod. Seven feet itself, whatever it may be, eludes the senses completely. Length, as opposed to objects having length, has a very general or abstract character. A length of seven feet is a property that is shared by all seven-foot beds, all seven foot oars, all seven-foot fishing rods, certain tall basketball stars and many other objects besides.

The things we have just listed have a useful characteristic: any two of them are equally long, as can be shown by comparing them to see if one overlaps the other when placed side by side. To claim for a certain bed and a certain oar that

bed is as long as oar (A)

is a statement whose truth or falsity can be decided by experimental means. In some respects statement (A) is like a statement of equality. We could abbreviate it as follows

$$\text{bed} \stackrel{L}{=} \text{oar}. \qquad (B)$$

The basic assertion made by (A) or (B) is that bed and oar are indistinguishable as far as length is concerned. In contrast, the statement

$$\text{bed} = \text{oar} \qquad (C)$$

claims that bed and oar are the same object. It is clearly false. Too much emphasis cannot be placed on correct use of the symbol =. To assert that A = B is to say that A and B are names for the same object -- nothing more.

The relations $\overset{L}{=}$ and =, though different in meaning, share three important characteristics. We name and describe them below in terms of $\overset{L}{=}$.

<u>Reflexive property</u>: $x \overset{L}{=} x$ for all objects x in the set.

<u>Symmetric property</u>: If $x \overset{L}{=} y$, then $y \overset{L}{=} x$.

<u>Transitive property</u>: If $x \overset{L}{=} y$, and $y \overset{L}{=} z$, then $x \overset{L}{=} z$.

You should satisfy yourself before proceeding further that these three statements are true for the relation $\overset{L}{=}$.

Many relations between objects in a set have these three properties. They are so numerous that the special name, <u>equivalence relation</u>, is used to classify them. It will be instructive to verify that of the six examples below, these first three are equivalence relations, and the last three are not.

(1) = , for the objects in any set.
(2) "lives in the same state as", for the set of U.S. residents.
(3) "has the same mother as", for the set of all students in your school.
(4) "is perpendicular to", for the set of all lines in a plane.
(5) "is the sister of", for the set of daughters of two given parents.
(6) "lives within 200 miles of", for the set of all California residents.

The idea of equivalence relation leads naturally to that of <u>equivalence class</u>. (Equivalence set would be just as good a term.) By the <u>length</u> equivalence class containing the seven-foot bed previously mentioned we mean the set of <u>all</u> objects that are precisely as long as the bed. It obviously contains much more than what we have chosen to list, but we would be inclined to say that every element in the class had a length of seven feet.

In a similar way we can contemplate volume equivalence classes, mass equivalence classes, time equivalence classes, and so on. Each equivalence class is defined in terms of an equivalence relation arising from a comparison experiment.

We are now prepared to say precisely what we mean by "seven feet". It is exactly that equivalence class containing all objects exactly as long as our (seven-foot) bed. By framing the definition in this strange way we succeed in making the concept "seven feet" refer to all objects seven feet long. We also settle clearly any question about whether length exists apart from the process of measurement. In this view every object has a unique length (namely, the equivalence class to which it belongs) even before we measure it. Finally, it is clear what we must be claiming when we assert equality of lengths. To say that

$$7 \text{ feet} = 84 \text{ inches}$$

means exactly what we wish the = sign always to say: that 7 feet and 84 inches are names for the same thing, our much discussed equivalence class containing the bed.

If this concept of length seems bizarre, it should be pointed out that no satisfactory explanation of what numbers are is any less intricate, and that the counting numbers can be defined satisfactorily only in terms of equivalence classes.

It might be best to not introduce the symbol $\stackrel{L}{=}$ to your classes. The degree of abstraction involved is rather high for students at this grade level. You can, however, point out that while bed = oar is clearly incorrect, we can write (length) bed = (length) oar and still maintain our traditional meaning for the equality.

Lengths (as well as other measurements) are extremely number-like. They can be <u>added</u>, in a sense of addition explained in Section 1.6, and they have order properties as shown in Section 1.12. They can also be multiplied by numbers. These traits account for their frequent confusion with numbers themselves.

In Section 1.1, <u>7 feet</u> is called a measurement (more specifically a length). 7 is termed a <u>measure</u> corresponding to the <u>unit</u> feet (or foot, if you will). It may help the teacher to be aware that the unit <u>feet</u> is itself a length and that the multiplication by 7 which seems to be indicated in the symbol <u>7 feet</u> is not familiar multiplication of number by number, but a new variety of multiplication of <u>number by length</u>. Its explanation appears in Section 1.6.

Exercise 1

1. Pick out the measures and the units in each of the following measurements. What might each measure?

	Measurement	Measure	Unit	Measurement of
(a)	8 acres	8	acre	a field
(b)	760 yards	760	yard	distance between two houses
(c)	27 lbs/in^2	27	lbs/in^2	pressure in a tire
(d)	11 fathoms	11	fathoms	water depth

Of course, the fourth column contains only possible suggestions. Students should come up with a wide variety of responses. It would be well worthwhile to discuss the practicality of their responses.

2. If a bathtub were filled by emptying a gallon bucket into it 30 times, what would be the volume of the bathtub? What is the measure? What is the unit?

   volume, 30 gallons
   measure, 30
   unit, gallon

3. The bathtub of Problem 2 is filled by using a quart container rather than a gallon bucket. (30 gallons = 120 quarts)

   (a) Does the volume remain the same? Yes
   (b) Is the measurement the same? Yes
   (c) What is the measure? 120
   (d) What is the unit? Quart

4. Change each of the following measurements to an equal measurement having a different measure and unit.

   (a) 3 minutes = 180 seconds = $\frac{1}{20}$ hour, etc.
   (b) 2 pounds = 32 ounces = $\frac{1}{50}$ cwt = $\frac{1}{1000}$ ton, etc.
   (c) 4 yards = 12 feet = 144 inches, etc.
   (d) 9 square feet = 1 square yard = 1296 square inches, etc.

1.3 Measurement

In Section 1.2 of the student text it is stated that measurement is a process of comparing some object or event to be measured with an appropriate unit of our choice. The authors feel that this definition should be brought into the discussion frequently until it seems to be understood and accepted by the students. Also, you should point out to your students that measurement involves some physical activity, i.e., experimentation. It might be helpful to have the students describe the physical activity that various measurements require, such as

- "weighing" an object
- "operating" a stop watch
- "using" a protractor
- "pacing off" a field:

The text gives two examples of mathematical models. The construction of a mathematical model depends so much upon the individual situation that it is virtually impossible to give a comprehensive definition, but the following steps are usually involved:

1. The first step in the construction of a mathematical model is to decide on the exact nature of the problem to be solved.
2. Interpret the physical situation in terms of numbers or mathematical relations.
3. Use your knowledge of mathematics to derive the needed mathematical information.
4. Predict the physical situation in light of the mathematical information.

The following examples may contribute to your understanding of mathematical models. You may find some of them useful in presenting this material to your students.

Example 1.

At certain stores, where there is a problem of knowing which customer should be served next, the customers might be asked to take their place in line, as at a ticket window, to assure being served in turn. A common method of handling such a problem in a more comfortable way is to issue a number to each customer as he comes in the store. Numbers will be issued in successive order. The merchant has thereby constructed a mathematical model of the queue. From this model he can make use of mathematics to determine several things:

(1) How many customers have been served;

(2) Which customer is next to be served;

(3) How many customers are waiting (by subtracting the number of the customer being served from the next number to be issued).

Example 2.

If we wish to determine the number of tiles needed to cover a hallway, we can employ a mathematical model of the hallway. We can make measurements of the hallway. We can now use our knowledge of mathematics to compute the area of the hallway, the area of a single tile, and then, the number of tiles needed to cover the floor.

Example 3.

A student might have a mathematical model of his father which would give him the necessary information needed when buying clothing for his father. The model might be as follows:

Hat size	=	$7\frac{1}{4}$
Neck size	=	16
Chest size	=	40
Sleeve length	=	34
Waist	=	36
Inseam	=	32
Shoe	=	11

Exercise 2

These questions are designed to provoke class discussion. The following are only suggestions and your students should make many more.

1. How would you find the length of your school building?

   Answer: Pace it off, use hand span, cubit measure (elbow to fingertips), measuring stick or tape.

2. What are several ways of timing a 50 yard race?

   Answer: Count slowly (for seconds), pulse beats, marching cadence, stop watch.

3. How could you weigh yourself if no scales were available?

   Answer: Get on a seesaw with someone who knows his weight, have a strong friend lift you and estimate your weight.

4. How could you compare the areas of two table tops if you had no ruler?

Answer: Cover with books and count the books, cover with notebook paper and count sheets. For a more accurate measure, use 3 × 5 note cards.

1.4 - 1.5 Measurement of Length: Ideas of Accuracy

The principal ideas in this section are that all measurements of length are to some extent crude and that our choice of unit depends on how crude or how fine a measurement we wish.

Exercise 3

1. Suggest suitable units for the following measurements.

(a) the altitude of an airplane;	10 ft, 50 ft, 100 ft, 1000 ft, mile
(b) the length of a car;	foot, inch, 1/2 inch
(c) the depth of the ocean;	fathoms, feet, miles
(d) the width of a window frame;	inch, 1/2 inch, 1/4 inch
(e) the width of a door frame;	inch, 1/2 inch, 1/4 inch
(f) the height of a truck;	foot, inch, 1/2 inch

2. What unit of measure would be acceptable when measuring the width of a window for drape rods? Inch

3. What unit of measure would be acceptable when measuring the width of a window to fit glass? 1/4 inch

4. What statement concerning choice of units of measurements is demonstrated by your answers to the questions above? Size of unit chosen depends on the amount of error we are prepared to ignore.

1.6 Addition of Lengths

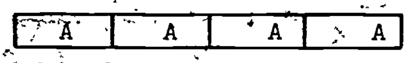
Knowing that lengths are equivalence classes, the teacher may wish to interpret length addition as equivalence class addition for his or her own benefit.

To form the sum of two equivalence classes, we select any element from the first and any element from the second and lay them end to end on a straight line with no overlap. (That is, we perform still a third kind of physical addition of objects.) The newly formed object has a length, or equivalence class to which it belongs. This third equivalence class is defined to be the sum of the two given equivalence classes.

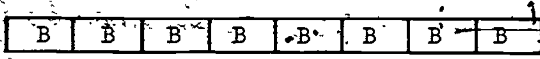
Multiplication of length by a counting number can now be defined by repeated additions as described in the text. Multiplication of lengths by other real numbers is deferred to Chapter 2.

Exercise 4

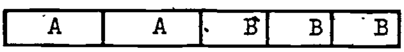
1. What is the total length in the following figures?

   (a) 

(4A)

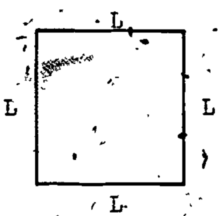
   (b) 

(8B)

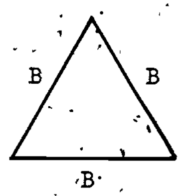
   (c) 

(2A + 3B)

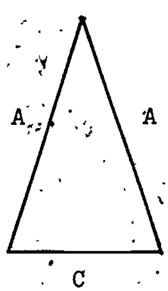
2. What is the perimeter of each of the following figures?

   (a) 

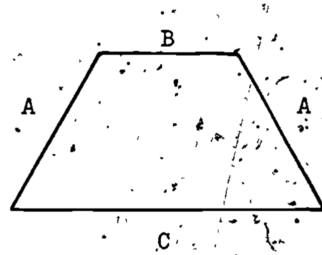
(4L)

   (b) 

(3B)

   (c) 

(2A + C)

   (d) 

(2A + B + C)

1.7 Unequal Numbers

Here the language and symbolism of inequalities are introduced for numbers. These facts are of general value in a mathematical education. The comparison property as stated in the student text is referred to by mathematicians as the principle of trichotomy.

Exercise 5

1. Use the symbols, " < ", " = ", " > ", to make a true statement in each of the following pairs of numbers.

(a) $5 < 6$

(b) $5 < 10$

(c) $16 = 16$

(d) $3 < 8$

(e) $19 > 11$

(f) $2 < 7$

(g) $9 > 4$

(h) $\frac{1}{4} = .25$

(i) $\frac{13}{15} > \frac{3}{4}$

(j) 7 ? a. Cannot be related without knowing the value of a.

1.8 More on Inequalities

The truth table is a device used by mathematicians to test the validity of a mathematical statement. In the student text we have discussed the use of the conjunctions "and" and "or" in joining two parts of a mathematical statement. Such statements may be represented symbolically by using the symbol " $\vee$ " for "or" and the symbol " $\wedge$ " for "and". The distinguishing feature of a mathematical statement, such as $5 < 3$, is that it is either "true" or "false".

A compound statement is formed by combining two or more simple statements. A symbolic statement made up of parts A and B and joined by the conjunction "or" would look like the following: $A \vee B$. There are four possible combinations of a two-part mathematical statement. The validity of the compound statement is dependent upon the validity of each of the simple statements. The compound statement $A \vee B$ is true or false as shown in this truth table.

A	B	$A \vee B$
T	T	T
T	F	T
F	T	T
F	F	F

The truth table associated with a compound statement using the conjunction "and" follows this pattern:

A	B	$A \wedge B$
T	T	T
T	F	F
F	T	F
F	F	F

Example:

Statement A --- Mary is a girl.

Statement B --- John is a girl.

Compound statement: Mary is a girl or John is a girl.

A	B	$A \vee B$
T	F	T

The compound statement is true. Notice that the "or" used in mathematical logic is the inclusive "or", sometimes symbolized "and/or".

Exercise 6

1. Rewrite each pair of inequalities below so that they are in the same sense. Write them as overlapping inequalities whenever possible.

(a) $3 < 4$, $3 < 5$

(b) $5 < 7$, $7 < 13$; therefore, $5 < 7 < 13$

(c) $16 < 21$, $19 < 21$

(d) $a < b$, $b < c$; therefore, $a \leq b < c$

(e) $25 < 31$, $21 < 30$

(f) $a < c$, $a < d$

(g) $n \leq m$, $m < 1$; therefore, $n < m < 1$

(h) $p < q$, $t \leq q$

(i) $7 < 16$, $4 \leq 7$; therefore, $4 \leq 7 < 16$

(j) $e < d$, $f < c$

2. What can be said of two numbers, a and b, if we know that $a \leq b$ and also, $a \geq b$? Explain your reasoning carefully.

Given $a \leq b$ so $a < b$ or $a = b$, and

$a \geq b$ so $a > b$ or $a = b$

to say $a < b$ and $a > b$ is a contradiction; therefore,

$a = b$.

3. Complete the following table by indicating in the proper space whether each part is true or false and whether the compound statement is true or false. (Last three columns are blank in students' text.)

Statement A	Conjunction	Statement B	A	B	Compound Statement
Example: $5 < 3$	and	$4 < 5$	F	T	F
(a) $5 < 3$	or	$4 < 5$	F	T	T
(b) $17 > 32$	and	$7 > 6$	F	T	F
(c) $(5-1) > (2+2)$	or	$(\frac{10}{2}) > (6-1)$	F	F	F
(d) $\frac{5}{7} < .72$	or	$.72 \leq \frac{5}{7}$	T	F	T
(e) $17 > 9$	and	$1\frac{5}{6} > 1.8$	T	T	T

4. Complete the following table by separating the compound statements into two parts. Tell whether each part is true or false, determine the conjunction which is indicated, and tell whether the compound statement is true or false.

Compound Statement	Statement A	Conj.	Statement B	A	B	Compound Statement
Example: $3 < 4 < 5$	$3 < 4$	and	$4 < 5$	T	T	T
(a) $5 \geq 4$	$5 > 4$	or	$5 = 4$	T	F	T
(b) $3 > 2 > 4$	$3 > 2$	and	$2 > 4$	T	F	F
(c) $6 \leq 6$	$6 < 6$	or	$6 = 6$	F	T	T
(d) $2\frac{3}{8} \geq 2.375$	$2\frac{3}{8} > 2.375$	or	$2\frac{3}{8} = 2.375$	F	T	T
(e) $\frac{13}{22} > \frac{30}{51} > \frac{40}{69}$	$\frac{13}{22} > \frac{30}{51}$	and	$\frac{30}{51} > \frac{40}{69}$	T	T	T

Lowest common denominator for (e) is $2 \times 3 \times 11 \times 17 \times 23$.

5. Answer true or false for each of the following.

(a) $5 \leq 5 \leq 6$ True

(b) $5 < 5 \leq 6$ False

(c) $5 \leq 5 < 6$ True

(d) $5 < 5 < 6$ False

1.9 Unequal Lengths

Although unequal numbers were not defined in the preceding sections, unequal lengths are carefully defined in the present one. Notice that the definition depends on the prior definition of addition for lengths, and in no way involves reference to numbers.

Exercise 7.

1. For lengths U, V and W, write in words the following statements.

(a) $U < V < W$ (length) U is less than (length) V and (length) V is less than (length) W.

(b) $U \leq V$ (length) U is less than or equal to (length) V.

(c) $U \leq V < W$ (length) U is less than or equal to (length) V and (length) V is less than (length) W.

2. Determine which statements are true and which are false. Give reasons in each case.

(a) 3 feet $\geq$ 39 inches False

3 feet = 36 inches and 36 inches < 39 inches

(b) 42 inches $\leq 3\frac{1}{2}$ feet $< 1\frac{1}{3}$ yards

True

$3\frac{1}{2}$ feet = 42 inches

$1\frac{1}{3}$ yards = 48 inches

42 inches = 42 inches < 48 inches

(c) 40 inches > 2 feet < 1 yard

False

It is true that 40 inches > 2 feet,

but 1 yard = 3 feet and 2 feet < 3 feet.

1.10 Measurement and Standard Units: A Classroom Experiment

The experiment described is intended to illuminate the way in which choice of unit affects the resulting measure. It also dramatizes the need for adopting standard units. In the course of this particular experiment, however, you should make the students feel that their own units are quite as good as any other, possibly a bit better. In this way you can convey the idea that there is no inherent virtue in standard units such as yards and meters. Only convenience in communication causes us to use them. You may want to personalize the students' units a bit by naming them for their owners. Thus "Bob" could be the name of the length of Bob's stick and it could be discovered, for instance, that

$$18 \text{ Bobs} \leq L < 19 \text{ Bobs}.$$

The following are suggested answers to the questions raised in the students' text.

1. You would most likely want to measure the length of a room to the nearest foot. Therefore, a stick about 1 foot long should be chosen.
2. The inequality reported by Don $(29 \leq d < 31)$ indicates he did not follow instructions.
3. (a) Bob had the smallest number of units in his measurement; therefore, he must have used the largest stick.
   (b) Frank had the smallest stick.
4. Don and Carla had sticks which were closest to each other in length.
5. No.
6. If everyone used the same size stick, all the inequalities should be the same.
7. Yes.
8. Yes, provided you carried the piece of chalk with you.
9. You would be able to describe your measurement in terms of a standard unit.

1.11 Standard Units

You may want to spend some class time discussing the importance of having standard-size units. In the past, most units of length were based on the human body. Have some of your students look up the history of the pace, the inch, the cubit, the yard, and the hand. Have your students measure the width of their palms and compare all of the results.

For a vivid demonstration, have a group of your students find and bring to class a length of threaded $\frac{3}{4}''$ steel pipe. Another group should be asked to bring a $\frac{3}{4}''$ pipe fitting. Show that even though these pieces came from different places, they will fit together well.

1.12 Length Measurement and Counting.

The procedure of length estimation described in this section depends on carefully formulated concepts of length addition (Section 1.6) and length inequality (Section 1.8). It will probably be helpful to use the data collected in the experiment on measurement by substituting numbers in the literal statements given in this section. Teachers testing these materials have suggested that the substitution of numbers in the literal statements help children to understand the generalizations.

1.13 Further Properties of Order: The Transitive Property

This section and the next are important because they extract the principal mathematical by-product of our study of measurement, deeper knowledge of order properties.

It is a fair assumption that students at this level have very weak intuitive ideas about order. The demonstrations described here are designed to build greater confidence in using the transitivity pattern and in so-called "addition" of inequalities.

It may be helpful in Section 1.13 to give the student a broader view of transitivity from your private knowledge. Many relations other than order relations are transitive. Examples of some that are and some that are not may help to get the point across. Such examples have been included in Exercise 8.

To stimulate ideas about length inequality you may find it helpful to have in the classroom about a dozen sticks ranging in length from about 15 inches to 21 inches in jumps of $\frac{1}{2}$ inch. Dowel stock, available in hardware stores in three-foot lengths, should serve well. Mark each stick on the end with its own letter of the alphabet, making sure that alphabetical order and length order are not the same.

Repetition of the following procedure many times will provide a useful basis for class discussion. Select three sticks at random and give one to each of three students designated as the first, the second and the third student. Let the first and second students compare their sticks; likewise the second and the third. Record the results on the blackboard in some such fashion as

$K > M$, $M > G$. (Emphasize that these are length inequalities.)

Invite the remaining students to predict an order relation between K and G, if they are able. Here $K > G$ is a plausible prediction. If it is made, an immediate verification is called for by comparing K and G directly.

Of course, many results will turn out to be in the following pattern,

$F > H$, $H < D$,

in which case no confident prediction can be made about an inequality between F (the first object) and D (the third object). Getting the students to use restraint where no prediction is plausible is just as important as inducing them to make a guess where one is warranted.

A simple variation of this exercise is to compare students' heights in pairs by having them stand back-to-back in the customary manner. A student's set of textbooks may also be used.

The goal of the demonstration is to develop in the students the willingness to formulate a very general statement (with your guidance, of course) in some such form as this: "Given any three sticks, if the first is longer than the second, and the second is longer than the third, then the first must be longer than the third." You should emphasize in the course of discussion that the conclusion reached is not a statement about mathematics, but is a statement about the physical hypothesis which demonstrates a pattern often encountered within mathematics. Also, you should point out that statements about sticks which are accepted as true do not prove anything about numbers, although they may suggest statements about numbers.

This same routine can be performed with other measurements, specifically mass and volume. Such exercise will pave the way for discussion of other magnitudes in later chapters. It will also emphasize that similar patterns arise in different contexts.

To compare masses a simple beam balance can be easily assembled, as suggested by the photographs. The beam, a meter stick or yardstick, hangs on a triangular file knife edge which passes through the holes of a paper clamp attached to the center of the stick. The file rests across two inverted Dixie cups. The pans are Dixie cups suspended from the ends.

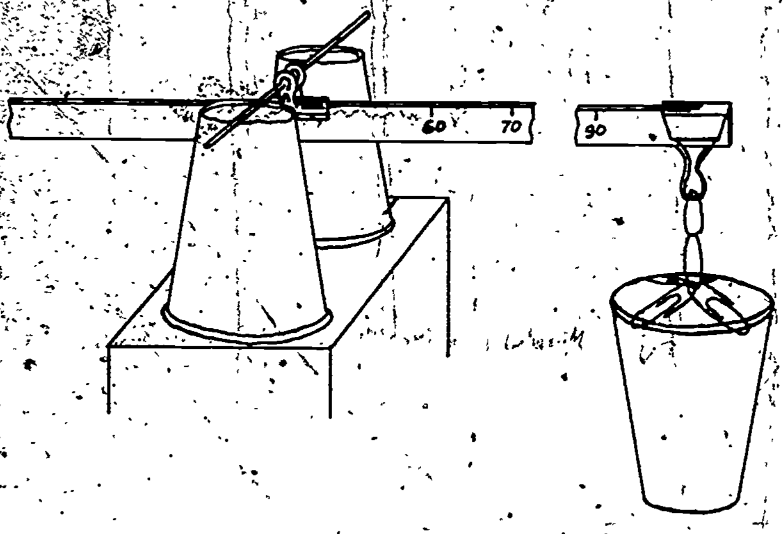


Figure 1

Again, to perform the demonstration, assemble a dozen or so small objects of comparable mass, and for three selected objects let the students compare the first with the second, then the second with the third. Record the results and invite class predictions where possible.

Exercise 8

1. State the transitive property of number inequality using "greater than".

   If a, b and c are real numbers, and if $a > b$ and $b > c$, then $a > c$.

2. Is there a transitive property for equality?

   Yes. If a, b and c are real numbers, and if $a = b$ and $b = c$, then $a = c$.

3. Test the following phrases for transitivity.

   (a) "is older than"

   Transitive. If "a is older than b" and "b is older than c", then "a is older than c".

   (b) "Lives within 200 miles of"

   Not Transitive. If "a lives within 200 miles of b" and "b lives within 200 miles of c", a does not necessarily live within 200 miles of c.

   (c) "has the same mother as"

   Transitive. If "a has the same mother as b", and "b has the same mother as c", then "a has the same mother as c".

(d) "is taller than"

Transitive. If "a is taller than b", and "b is taller than c", then "a is taller than c".

(e) "lives next door to"

Not Transitive. If "a lives next door to b", and "b lives next door to c", a does not necessarily live next door to c.

4. a, b, c and d are numbers. State an inequality between a and d (whenever possible) in each of the following cases.

(a) $\underline{b < a}$, $c < b$, $\underline{b > d}$ thus $d < a$

(b) $\underline{d > b}$, $b < c$, $\underline{b > a}$ thus $d > a$

(c) $\underline{d < c}$, $\underline{a > c}$, $b \geq a$, $d < b$ thus $d < a$

(d) $c > b$, $\underline{a > b}$, $\underline{b > d}$ thus $a > d$

(e) $a < c$, $b < c$, $d > b$ No conclusion; cannot form an overlapping inequality involving a and d (c may be greater than d)

5. In the four exercises below all letters refer to numbers.

(a) If $p < q$ and $q < r$, does it follow that $r > p$?

Yes, by the transitive property.

(b) If $m > p$ and $p > n$, does it follow that $m > n$?

Yes, by the transitive property.

(c) If $d > f$ and $d > g$, does it follow that $f > g$?

No. It does not form an overlapping inequality.

(d) If $h < k$ and $j < h$, does it follow that $k > j$?

Yes, by the transitive property $j < k$ which is equivalent to $k > j$.

1.14 Further Properties of Order: Addition

The equipment of the preceding section can be used again to good advantage in creating confidence about a new pattern, namely, the addition of inequalities.

Experimentation with sticks will result in the accumulation of many facts, such as: $K < G$ and $D < H$. Focus the class attention on a pair of such inequalities and ask for opinions about a relation between $K + D$ and $G + H$. Here $K + D$ is a name for the new stick created by laying K and D end to end, similarly for $G + H$. (You may find it useful to write the symbols $K \oplus D$ and $G \oplus H$ to emphasize that a new kind of addition is involved.) While opinions are being formed, you should carefully conceal objects K, D, G and H, and force the students to use only the two given inequalities in formulating a guess.

When the desired prediction,

$$K + D < G + H$$

is made, immediate verification should be made by experiment. Repeat the routine with another pair of inequalities until you think you have established sufficient confidence. Then ask the students to generalize their results. Point out that any conclusions drawn do not <u>prove</u> anything about numbers, but do suggest something about lengths.

Vary the experiment to your taste by adding mass inequalities. "Adding" objects for mass purposes merely means putting them together on the same pan of the balance.

Exercise 9

1. Given the four numbers, R, S, T, U where R > S and T > U. Prove that

$$R + T > S + U$$

Suggestion: add T to both numbers of R > S and add S to both numbers of T > U. Now apply the transitive property to the two new statements.

(1)	$R > S$	given
(2)	$R + T > S + T$	addition property of number inequality
(3)	$T > U$	given
(4)	$S + T > S + U$	addition property of number inequality
(5)	$R + T > S + U$	Statements 2 and 4 form an overlapping inequality and the transitive property may be applied.

2. What can be said about the sum of numbers x and y if

(a)
$$\begin{array}{l} 2.2 < x < 2.5 \\ \underline{4.1 < y < 4.5} \\ 6.3 < x+y < 7.0 \end{array}$$

(b) First rewrite $3 > y > 1$ to read $1 < y < 3$

$$\begin{array}{l} 1 < y < 3 \\ \underline{16 < x < 18} \\ 17 < x+y < 21 \end{array}$$

(c) First rewrite $x > 9$ as $9 < x$

$$\begin{array}{l} 11 < y \\ \underline{9 < x} \\ 20 < x+y \end{array}$$

(d) Nothing can be said about the sum of x and y.

3. Suppose that a, b, c and d are numbers.

(a) If $a < b$ and $c \leq d$, how do the sums $a + c$ and $b + d$ compare?

$a + c < b + d$

(b) If $a \leq b$ and $c \leq d$, how do the sums $a + c$ and $b + d$ compare?

$a + c \leq b + d$

Sample Test Items

1. For each of the following measurements
   A. Identify the measure and the unit
   B. Change each to an equal measurement using a different unit.
   (a) 147 pounds
   (b) $1\frac{1}{2}$ hours
   (c) 5 feet
   (d) 6 quarts

2. Use one of the symbols, " < ", " = ", " > ", to make a true statement for each of the following pairs.
   (a) 10, 7
   (b) 3 feet, $\frac{11}{17}$ yard
   (c) 5 minutes, 350 seconds
   (d) (5 + 1), $\frac{36}{6}$
   (e) 7, c

3. Rewrite each pair of inequalities below so that they are in the same sense. Write them as overlapping inequalities wherever possible.
   (a) $7 > 6$, $7 < 9$
   (b) 2 feet < 1 yard, $\frac{1}{2}$ foot < 24 inches
   (c) $p < q$, $q > t$
   (d) $25 \leq 27$, $25 > 19$
   (e) 3 hours < 1 day, 1 week > 24 hours

4. Assume that U, V, W are lengths, and select the greatest length in each of the following statements, if possible.
   (a) $U < V < W$
   (b) $U \leq V$, $V > W$
   (c) $U > V$, $V \geq W$
   (d) $U < V$, $V > W$
   (e) $U > V > W$

5. With the paper provided used as a unit measure the length and width of this paper and express the measurement as an inequality.

________________ = length

________________ = width

Note: Supply a small piece of construction paper as a standard unit of length.

6. Classify each of the following relations as transitive or not transitive.

(a) "divides evenly into"

(b) "costs 10¢ more than"

(c) "costs more than"

(d) "is the uncle of"

7. W, X, Y and Z are numbers. State an inequality between W and Z in each of the following cases.

(a) $W > Y, X > Z, Z < Y$ ________________

(b) $W < X, X > Y, Z > X$ ________________

(c) $Z < Y, W > Y, X > W, Z < X$ ________________

8. What can be said about the sum of numbers M and N if:

(a) $23 < M < 26$ and $42 > N > 35$ ________________

(b) $27 > M$ and $N < 13$ ________________

(c) $M \leq 11$ and $54 \geq N$ ________________

(d) $M < 32$ and $42 < N$ ________________

9. Complete the following statements accurately.

When two numbers are compared as to size, there are three possible results:

(a) the numbers may be ________________

(b) the first may be greater ________________

(c) the first may be ____________ the second.

10. If three different people, Ann, Roger, and Cathy chose different units to measure the length of a room and came up with these inequalities of measure, which person chose the largest unit of measure?

Ann	$34 \leq m < 35$
Roger	$29 \leq m < 30$
Cathy	$41 \leq m < 42$

11. Using the numbers a, b, c and d

(a) Give an example of a pair of inequalities in the same sense. ____________

(b) Give an example of a pair of inequalities in the opposite sense. ____________

(c) Give an example of a pair of inequalities in the same sense which could be written as an overlapping inequality.

____

12. True or False. Circle T if you think the statement is true. Circle F if you think the statement is false.

(a) $6 \leq 9 < 7$ T F

(b) $5 \leq 5 < 9$ T F

(c) $9 \leq 6 < 7$ T F

(d) $8 > 7 > 6$ T F

(e) $9 \leq 10 < 11$ T F

*13. Complete the following table.

	Statement A	Conj.	Statement B	Compound Statement (if possible)	A (True or False)	B (True or False)	Compound Statement (True or False)
(a)	$5 < 6$	and	$6 < 7$				
(b)				$5 \leq 3$			
(c)				$5 < 7 < 6$			
(d)				$15 \geq 7$			
(e)				$7 \leq 9$			
(f)	$5 < 3$		$5 < 6$				T
(g)	$10 < 12$		$15 < 2$				F

Sample Test Items -- Answers

1.

	A.		B.
	Measure	Unit	Measurement
(a)	147	pound	2,352 ounces
(b)	$1\frac{1}{2}$	hour	90 minutes
(c)	5	foot	$1\frac{2}{3}$ yards or 60 inches
(d)	6	quart	12 pints or $1\frac{1}{2}$ gallons

2. (a) $10 > 7$

(b) 3 feet $> \frac{11}{17}$ yard

(c) 5 minutes $<$ 350 seconds

(d) $(5 + 1) = \frac{36}{6}$

(e) 7 cannot be related to c unless value for c is given.

3. (a) $6 < 7,\ 7 < 9 \quad \therefore\ 6 < 7 < 9$

(b) 1 yard $>$ 2 feet, 24 inches $> \frac{1}{2}$ foot

1 yard $>$ 2 feet $> \frac{1}{2}$ foot

or 1 yard $>$ 24 inches $> \frac{1}{2}$ foot

(c) $p < q,\ t < q$ (not overlapping)

(d) $27 \leq 25,\ 25 > 19 \quad \therefore\ 27 \geq 25 > 19$

(e) 3 hours $<$ 1 day, 24 hours $<$ 1 week

$\therefore$ 3 hours $<$ 1 day $<$ 1 week

3 hours $<$ 24 hours $<$ 1 week

4. (a) W

(b) Can't tell; U may be equal to V.

(c) U

(d) V

(e) U

5. Should be of the form: $bU \leq m < (b + 1)\ U$

Size of "b" depends on size of unit you issue.

6. (a) transitive

(b) not transitive

(c) transitive

(d) not transitive

7. (a) $Z < W$

(b) $W < Z$

(c) $Z < W$

36

8. (a) $58 < M + N < 68$

   (b) $M + N <$ [illegible]0

   (c) $M + N \leq 65$

   (d) Nothing can be said about the sum of M and N

9. (a) Equal

   (b) than the second

   (c) less than

10. Roger

11. (a) $a < b,\ c < d$

    (b) $a < b,\ c > d$

    (c) $a < b,$ [illegible] $< c$

12. (a) False

    (b) True

    (c) False

    (d) True

    (e) True

*13.

	Statement A	Conj.	Statement B	Compound Statement (if possible)	A (True or False)	B (True or False)	Compound Statement (True or False)
(a)				$5 < 6 < 7$	T	T	T
(b)	$5 < 3$	or	$5 = 3$		F	F	F
(c)	$5 < 7$	and	$7 < 6$		T	F	F
(d)	$15 > 7$	or	$15 = 7$		T	F	T
(e)	$7 < 9$	or	$7 = 9$		T	F	T
(f)		or		$5 < 3$ or $5 < 6$	F	T	
(g)		and		$10 < 12$ and $15 < 2$	T	F	

Chapter 2

LENGTH AND THE NUMBER LINE

2.1 Using Related Units in Measuring

This chapter begins with the Unmarked Stick Experiment to illustrate the problems involved in measurement, and ends with a discussion of the number line. Other important topics treated are the metric system of length, the decimal approximations to a real number, the meaning of measurement to the nearest unit, exponents and scientific notation.

The experiment should bring out the following mathematical ideas:

1. The extension of the number system from the counting numbers (positive integers) of Chapter 1, when you placed units end-to-end, to the rational numbers and other real numbers (irrational numbers) by the constructions of Chapter 2.

2. The constructions use the following statement from geometry:

   If a set of parallel lines cuts off equal segments on one transversal, it cuts off equal segments on every other transversal. This fact is never explicitly stated but it is used in the process of dividing the line. (See Figure 2.)

3. A real number may be defined in advanced mathematics, but we usually use rational approximations to represent real numbers. ($\doteq$ means approximately equal to.) For example, $\sqrt{2} \doteq 1.4142$, $\sqrt{3} \doteq 1.732$, $\pi \doteq 3.1416$, $\frac{1}{12} \doteq .08333$, $\frac{1}{11} \doteq .090909$, $\frac{1}{2} \doteq .4999$. More precisely, rational numbers are represented by repeating or terminating decimals while other real numbers are represented by non-repeating, non-terminating decimals.

4. The definition of positive, negative and zero exponents as powers of 10 and the laws of exponents for powers of 10.

5. A length can be measured in many units by using appropriate conversion factors. The measure of the length in terms of one unit can be transformed into the measure of the length in terms of another unit.

2.2 Unmarked Stick Experiment

Students are to work individually. Each student should have the following equipment:

1 straight stick (approximately 15")
2 sheets of notebook paper (lined)
1 sheet of graph paper (10 squares per inch)
1 ball of string } for entire class
1 small roll of masking tape } for entire class

The purpose of this experiment is:

1. To show that the choice of unit is arbitrary. In addition, the choice of the number of equal divisions of any unit is arbitrary. We use a decimal system since our number system (Hindu-Arabic) has a base ten. Our monetary system is decimal. The metric system which we find used on the calibration of film sizes, foreign cars, some rifles, and in the Olympic Games is a decimal system.

2. To obtain upper and lower estimates for the length measure of different objects.

3. To learn how to divide a length into equal parts.

4. To obtain a succession of better estimates of a length.

5. To compare different student's estimates of the same length.

6. To reinforce the value of the decimal system.

7. To emphasize the need for standard units.

8. To discuss the conversion of units.

9. To discuss errors in measurement due to human variability.

Details of the experiment:

Each student must bring to the classroom a straight-edged unmarked stick approximately 15 inches long. The student should use his stick to measure the length of one of his books. He should record the result both as an inequality and as an estimate to the nearest unit, as illustrated in Table 1 in 2.2 of student text. He should make a similar measurement and a record of the height of the door and of the width of the classroom.

Sample Data

Table I

$0 \leq$ measure of book < 1, nearer to 1
$4 <$ measure of door < 5, nearer to 5
$15 <$ measure of room < 16, nearer to 15

All measures in U, while neighbor's unit = 1.12U.

He should take two sheets of ruled notebook paper and check that the lines are equally spaced by comparing the distance between lines on one sheet with the distance between one pair of lines on the other sheet as illustrated in Figure 3 of the student text.

After deciding that the lines are equally spaced, the student should scotch-tape the two sheets together as illustrated in student text Figure 4.

Mark the topmost line with a zero and then mark every third line consecutively with a numeral from 1 to 10. Place the stick so that one end is lined up with the line marked 0 and the other end of the stick is lined up with the line marked 10. Using the numerals from 1 to 9 inclusive, mark the points where the corresponding marked lines meet the stick.

The experiment should be done in stages. After the data in Table I has been recorded, it is desirable to have some class discussion about the inadequacy of this measuring instrument and the need for division. It should be emphasized that, for the time being, we do not have any standard measuring equipment and that we do not have any mechanical means for dividing lengths into equal parts. We do not know that the lines on the note paper are equally spaced. This will be checked in Step 2 of the experiment. If we did not have ruled paper, we could construct some. Make two marks on a stick an arbitrary distance apart. Using this distance lay off a succession of marks on one edge of the paper. We do the same on the other edge of the paper and then connect corresponding lines as follows:

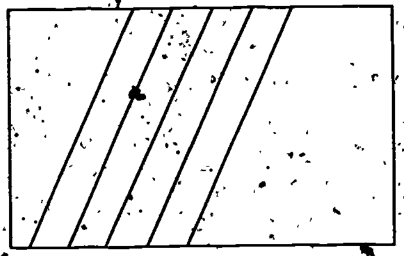


Figure 1

Emphasize that the lines need not be perpendicular to the edge of the paper; they need only to be equally spaced.

Before the stick is divided into 10ths, there should be a class discussion of Figure 2. Begin with a triangle ABC. Then divide the side AC and side BC into four equal parts. Draw lines through the corresponding divisions and point out that the lines are parallel. If another line, not parallel to the base, is drawn anywhere in the plane the parallels will cut off three equal segments on it. See the figure below.

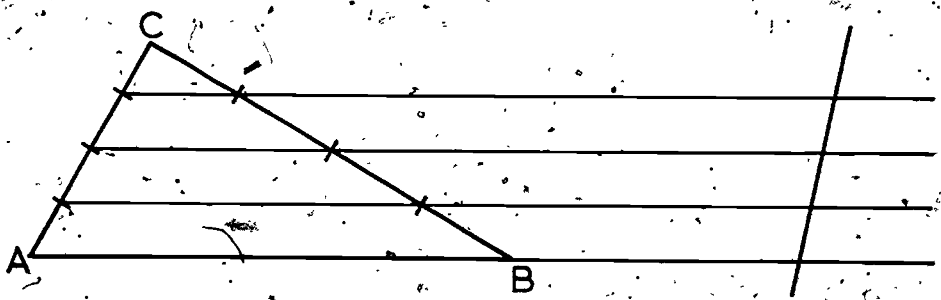


Figure 2

Use the stick marked into 10ths to again measure the length of a book, the height of a door, and the width of the room. Record the data in Table II. (Teacher may stop here with some groups.)

Sample Data

Table II

$0.6 <$ measure of book < 0.7, nearer to 0.7
$4.8 <$ measure of door < 4.9, nearer to 4.8
$15.3 <$ measure of room < 15.4, nearer to 15.3

All measures in U, while neighbor's unit = 1.12U.

Use a sheet of graph paper containing 10 squares to the inch and number successive lines on it from 0 to 10 inclusive. Place one end of the stick on the line marked 0. Adjust the stick so that the point marked 1 on the stick coincides with the line marked 10 on the sheet of graph paper. Mark the stick at each point where the numbered lines intersect it. However, the marks should not be numbered. This procedure has divided the first 10th of the stick into 100ths. Continue this procedure until the entire stick has been divided into 100ths.

Repeat the measurement of the book, the door and the room. Record your measurements in Table III.

Sample Data

Table III

$0.68 <$ measure of book < 0.69, nearer to 0.68	
$4.83 <$ measure of door < 4.84, nearer to 4.84	
$15.31 <$ measure of room < 15.32, nearer to 15.32	

All measures in U, while neighbor's unit = 1.12U.

For later use, each student should measure the length of his neighbor's stick to the nearest 100th of his unit.

The student should be ready to make a generalization of the method used in dividing a line into an arbitrary number of equal parts. Suppose we want to divide a line AB into four equal parts. From point A draw a line in an arbitrary direction and lay off five equally spaced, but arbitrary, lengths on it. Call the end point of the fifth length C. Through C draw a line in an arbitrary direction. Connect B with the fourth marked point on the line AC, and let D be the point of intersection of this line with the new line through C. Lay off the length CD three more times on the line CD. Connect corresponding points on the lines CD and AC. These lines will divide the line AB into four equal parts. See Figure 3.

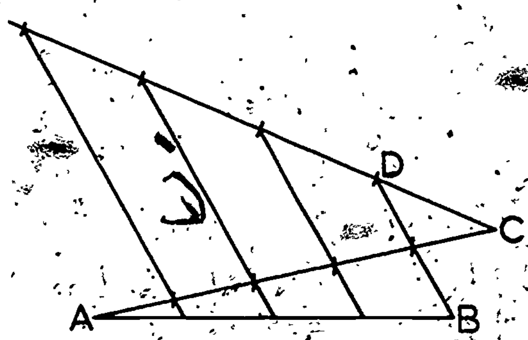


Figure 3

After the student has completed his three sets of measurements, a classroom discussion about the difficulty in comparing the measurements of different students leads naturally to the need for a standard unit and to the introduction of the metric system.

Teachers desiring to use the technique of the Unmarked Stick Experiment for other objects or events may use the Soda Straw Balance Experiment found in the Physical Science Study Committee, Physics, Laboratory Guide, Heath, 1960, page 11.

Exercise 1

1. Explain why moving the sheets along each other as indicated in Figure 3 shows that the lines are equally spaced.

   If the lines were not equally spaced, they would not coincide or match.

2. How do you know that your procedure for dividing the stick into ten parts of equal length really works? Devise a method to divide the stick into seven equal parts.

   If the lines are equally spaced, then any other line drawn across these lines is divided into equal line segments by these lines. Place the stick so that the ends of the stick are on lines marked 0 and 7, then mark on the stick where the lines marked 1 through 6 meet the stick.

3. If you are measuring your desk with a meter stick marked in divisions down to millimeters, how many decimal places will you be able to give in the measure inequality if the measure is based on units of

   (a) Meters?

   Answer: 3

   (b) Centimeters?

   Answer: 1

   (c) Kilometers?

   Answer: 6

4. Measure the length of your neighbor's stick with your stick, and find its measure to the nearest hundredth of your unit. Use this number to convert his data in Table I to measures in terms of your unit. Do the results you find this way agree with your measurements noted in Table I? If the agreement is unsatisfactory, can you give some explanation for the disagreement?

5. Convert your neighbor's data in Table III to measure in terms of your unit. Do these results agree with your measurements noted in Table III? Is the agreement between your results and your neighbor's results better or worse now than in Problem 5?

4 and 5. Let us call the length of your stick 1U, and call the length of your neighbor's stick 1V, and then we find that

$$1V = 1.12U. \tag{1}$$

Then if he measures the width of the room as 1.26V, we can convert to U units as follows:

$$1.26V = 1.26 \times (1.12U) = 1.4112U.$$

Since the neighbor's length was measured only to the nearest 100th of a unit, this result should be rounded off to 1.41U.

Notice also that from (1) we obtain

$$1U = \frac{1}{1.12} V = 0.89V$$

by dividing by 1.12 (or multiplying by $\frac{1}{1.12}$) so that it is possible to convert U-measures into V-measures.

Say the desk measures 1.37U; then

$$1.37U = 1.37 \times (0.89V) = 1.2193V$$

and we round off to 1.22V .

2.3 The Metric System of Length

As stated in the first purpose of 2.2, the choice of unit is arbitrary and the choice of equal divisions is arbitrary, but the metric system provides us with a variety of units each of which is divided in ten equal divisions. Therefore, it is a decimal system. In the student text, we have limited the discussion to the metric system of length. The teacher may feel he needs to include other objects and events such as area, volume or capacity, weight or mass, density and velocity. For example, the "glugs" or "fish weights" used in Chapter 4 of the text could be used for arbitrary units of capacity and mass respectively.

Students should become familiar with the prefixes used in the metric system. They should be directed to various sources of reference for their information.

Notice that in this text a length of one meter is not the length of one particular distance but is a property of all sticks, or objects, or distances which have the length one meter. Therefore, a statement such as

$$1 \text{ meter} = 100 \text{ centimeters}$$

is an actual equality, since one meter and one hundred centimeters are both names for the same length. A length such as 3.6 meters means a length which is 3.6 times a length of 1 meter. That is why we can write

$$3.6 \text{ m} = 3.6 \times (1\text{m}) = 3.6 \times (100 \text{ cm}) = 360 \text{ cm}$$

and 100 cm is therefore the conversion factor to change from meters to centimeters. Similarly, since

$$1 \text{ centimeter} = \frac{1}{100} \text{ meter}$$

and

$$360 \text{ cm} = 360 \times (1 \text{ cm}) = 360 \times (\frac{1}{100} \text{ m}) = 3.6 \text{ m}$$

the conversion factor to change from centimeters to meters is $\frac{1}{100}$ m. Problems 4 and 5 of Exercise 1 show this conversion for the student and his neighbor's units. This procedure can be extended to other systems as the problems indicate.

Exercise 2

1.

	Millimeter Measure	Centimeter Measure	Decimeter Measure	Meter Measure
a	56720	5672	567.2	56.72
b	2047	204.7	20.47	2.047
c	.056	.0056	.00056	.000056
d	2300	230	23	2.3

2.

	Dekameter Measure	Hectometer Measure	Kilometer Measure
a	5.672	.5672	.05672
b	.2047	.02047	.002047
c	.0000056	.00000056	.000000056
d	.23	.023	.0023

3. Describe a counting process that could lead to a meter-measure of 23.987.

By counting 23 meters, 9 decimeters, 8 centimeters, and 7 millimeters.

4. 5 kilometers = 5000 meters

256 millimeters = .000256 kilometers

1245 centimeters = 12450 millimeters

536 centimeters = 5.36 kilometers

5. $\frac{1}{12}$ foot = 1 inch

1760 yards = 1 mile

1 Angstrom = $\frac{1}{100,000,000}$ centimeter

6 feet = 1 fathom

$16\frac{1}{2}$ feet = 1 rod

$\frac{1}{2.54}$ inch = 1 centimeter

6. (a) 12 feet = 144 inches

(b) $\frac{1}{88}$ mile = 20 yards

(c) 12,000 fathoms = 72,000 feet

(d) 1,000 Angstroms = $\frac{1}{10,000}$ meter

(e) 1 meter $\doteq$ 3.24 feet [$1 \text{ m} = 100 \text{ cm} = 100 \times (\frac{1}{2.54} \text{ in}) = 100 \times (\frac{1}{2.54}) \times (\frac{1}{12}) \text{ ft}$]

(f) 1 inch = 254,000,000 Angstroms [1 in = 2.54 cm = 2.54 × (100,000,000 Angstroms)]

2.4 and 2.5 . Successive Approximations to a Length Measure and The Determination of Length Measure

The aim of these two sections is to show the students that a sequence of lower and upper estimates can be obtained for the measure of any length, and to convince him that this sequence defines a number. The text intentionally ignores the distinction between rational and irrational numbers. It may be desirable, with some students, to point out the distinction to show that every rational number has a periodic decimal expansion. Conversely, every periodic decimal represents a rational number. Finally, show that a nonterminating, nonperiodic decimal is an irrational number, and conversely. Useful material for this topic can be found in SMSG Mathematics for Junior High School, Volume 2, Part I, Chapter 6, or Irving Adler, The New Mathematics, Mentor, 1960.

Exercise 3

1. Which of the following is a more accurate estimate for the same length?

   (a) $3.25 \leq d < 3.26$ in meter measure or
   $3256 \leq d < 3257$ in millimeter measure.

   $3256 \leq d < 3257$ in millimeter measure

   (b) $561 \leq d < 562$ in centimeter measure or
   $56 \leq d < 57$ in millimeter measure.

   $561 \leq d < 562$ in centimeter measure

   (c) $4789 \leq d < 4790$ in millimeter measure or
   $4.789 \leq d < 4.790$ in meter measure.

   both the same

2. Which of the following gives the most accurate estimate of a certain length?

   (a) in meter measure $5.81 \leq d < 5.82$
   (b) in kilometer measure $0.0058 \leq d < 0.0059$
   (c) in millimeter measure $5811 \leq d < 5812$
   (d) in centimeter measure $581.14 \leq d < 581.15$

   (d) is the only correct response

3. Suppose the dekameter measure m of a length satisfies the inequality $8.9674 \leq m < 8.9675$. Write inequalities for m in terms of the units used in Problem 2.

   (a) in meter measure $89.674 \leq m < 89.675$
   (b) in kilometer measure $0.089674 \leq m < 0.089675$
   (c) in millimeter measure $89674 \leq m < 89675$
   (d) in centimeter measure $8967.4 \leq m < 8967.5$

4. At what stage in the following series of approximations has the procedure of successive subdivision been violated?

$$5 \leq d < 6$$
$$5.6 \leq d < 5.8$$
$$5.63 \leq d < 5.64$$
$$5.631 \leq d < 5.632$$

   $5.6 \leq d < 5.8$

   Does any one of these inequalities contradict another?

   No.

Exercise 4

1. Write down the first five successive estimate inequalities in decimal form, to the following length measures:

(a) 49.3747921

$$49 \le m < 50$$
$$49.3 \le m < 49.4$$
$$49.37 \le m < 49.38$$
$$49.374 \le m < 49.375$$
$$49.3747 \le m < 49.3748$$

(b) 8.999999

$$8 \le m < 9$$
$$8.9 \le m < 9.0$$
$$8.99 \le m < 9.00$$
$$8.999 \le m < 9.000$$
$$8.9999 \le m < 9.0000$$

(c) 3.22

$$3 \le m < 4$$
$$3.2 \le m < 3.3$$
$$3.22 \le m < 3.23$$
$$3.220 \le m < 3.221$$
$$3.2200 \le m < 3.2201$$

(d) 4

$$4 \le m < 5$$
$$4.0 \le m < 4.1$$
$$4.00 \le m < 4.01$$
$$4.000 \le m < 4.001$$
$$4.0000 \le m < 4.0001$$

(e) $4\frac{1}{2}$

$$4 \le m < 5$$
$$4.5 \le m < 4.6$$
$$4.50 \le m < 4.51$$
$$4.500 \le m < 4.501$$
$$4.5000 \le m < 4.5001$$

(f) $4\frac{1}{7}$

$$4 \leq m < 5$$
$$4.1 \leq m < 4.2$$
$$4.14 \leq m < 4.15$$
$$4.142 \leq m < 4.143$$
$$4.1428 \leq m < 4.1429$$

2. Guess the value of m in each of the following cases:

(a) $4 \leq m < 5$

$4.1 \leq m < 4.2$

$4.16 \leq m < 4.17$

$4.166 \leq m < 4.167$

and so on

$4\frac{1}{6}$

(b) $2 \leq m < 3$

$2.6 \leq m < 2.7$

$2.66 \leq m < 2.67$

$2.666 \leq m < 2.667$

and so on

$2\frac{2}{3}$

(c) $3 \leq m < 4$

$3.1 \leq m < 3.2$

$3.14 \leq m < 3.15$

$3.141 \leq m < 3.142$

$3.1415 \leq m < 3.1416$

$3.14159 \leq m < 3.14160$

and so on

π

Of which of your answers are you certain?

None.

2.6 How Lengths Are Quoted in Practice

Many students confuse "rounding off" a number when the teacher specifies the number of significant figures one time and the number of decimal places another time. The scientific notation coming up will overcome this difficulty. The problem posed when the last digit is 5 does not appear here because we are rounding off with the help of an inequality. The inequality specifies whether the given number is closer to the number below or the number above. For example, the student is not asked to round off a number such as 2.75 but, instead, to give the two-digit figure nearest to m where we know that

$$2.75 \leq m < 2.80 .$$

Clearly, the answer in this case is 2.8 . If the inequalities are not given and you are pressed to round off 2.75, it is suggested that you round to the next highest unit (in this case tenth); thus, the rounded value is 2.8 .

Exercise 5

1. How many significant figures are there in each of the following numbers?

(a) 573.02	5	
(b) 2.91	3	
(c) 3.14159	6	
(d) .005706	4	
(e) 5,296,000	4	(possibly 7)
(f) 3.760	4	(possibly 3)

2. Quote the following measures to two significant figures:

(a) $34.06 \leq m < 34.07$	34
(b) $.0765 \leq m < .0766$	.077
(c) $1374 \leq m < 1375$	1400
(d) $.000567 \leq m < .000568$	.00057
(e) $29,783 \leq m < 29,784$	30,000
(f) $125 \leq m < 135$	130
(g) $.0195 \leq m < .0205$	.020

3. Write the inequalities implied by the following measure statements: m is approximately.

(a) 2.6	$2.55 \leq m < 2.65$
(b) .075	$.0745 \leq m < .0755$
(c) 2604	$2603.5 \leq m < 2604.5$
(d) 1.059	$1.0585 \leq m < 1.0595$
(e) .003	$.0025 \leq m < .0035$
(f) 276.53	$276.525 \leq m < 276.535$

2.7 Exponents

Exponential notation for powers of ten is introduced. It is recommended that in the beginning only powers of ten be used. Notice that the definition of an exponent is changed slightly. The new definition has the advantage that a zero exponent has a natural meaning and that, later on, negative exponents will fit in quite naturally.

10^n where n is a positive integer is defined $\underbrace{10 \times 10 \times \ldots}_{\text{n factors}}$

$10^n \times 10^m$ is defined 10^{n+m}. Therefore,

$$10^n \times 10^{(-n)} = 10^{n+(-n)} = 10^0$$

and

$$10^n \times \frac{1}{10^n} = \frac{10^n}{10^n} = 1$$

then

$$10^0 = 1$$

Exercise 6

1. Write these numbers in a form that does not contain exponents.

(a)	63.475×10^2	6347.5
(b)	2.3×10^4	23,000
(c)	0.0004×10^3	0.4
(d)	10.562×10^0	10.562
(e)	10^6	1,000,000
(f)	3.008×10^1	30.08
(g)	16.0×10^3	16,000
(h)	5.280×10^4	52,800
(i)	10×10^2	1000
(j)	$1 \times 10^2 \times 10^3$	100,000

2. Write these numbers in a form using exponents in such a way that the decimal point follows the first significant figure.

(a)	6,400,000	6.4×10^6
(b)	6,475	6.475×10^3
(c)	4.56	4.56×10^0
(d)	314,159	3.14159×10^5
(e)	3,000,000,000	3×10^9
(f)	256	2.56×10^2
(g)	9,327.560	9.327560×10^3
(h)	98.763	9.8763×10

3. Which of the following are correct?

(a) $4 \times 10^3 \times 5 \times 10^2 = 20 \times 10^5$

(b) $7 \times 10^1 \times 4 \times 10^5 = 11 \times 10^6$

(c) $2 \times 10^3 \times 3 \times 10^2 = 6 \times 10^5$

(d) $19 \times 10^0 \times 7 \times 10^3 = 133 \times 10^3$

(e) $10^2 \times 10^3 \times 10^0 = 10^5$

(f) $10^2 \times 10^3 \times 10^1 = 10^5$

(g) $3 \times 10^2 \times 5 \times 10^3 = 1.5 \times 10^5$

(h) $1.2 \times 10 \times 1.2 \times 10^3 = 1.44 \times 10^5$

(i) $10^0 \times 2 \times 10^0 = 2 \times 1$

(j) $11 \times 10^2 \times 2.56 = 2.816 \times 10^4$

(k) $160 \times 2.5 \times 10^0 = 4 \times 10^2$

(l) $5 \times 10 \times 2 \times 10 = 10^3$

(a), (c), (d), (e), (i), (k), (l)

Note: These answers are numerically correct. Some answers are not written in scientific notation.

4. Fill in the blanks.

(a) 2.345 km = <u>234,560 cm</u>

(b) 2.4 km = <u>2400 m</u>

(c) 0.000064 m = <u>0.0064 cm</u>

(d) 5.62 cm = <u>0.0562 m</u>

(e) 37.6 mm = <u>0.0000376 km</u>

(f) 37.76×10^2 cm = <u>37,760 mm</u>

(g) $.057 \times 10^2$ m = <u>5700 mm</u>

(h) 3,762,598 mm = <u>3.762598 km</u>

(i) 4.578×10^5 cm = <u>4.578 km</u>

(j) 4.67×10^6 m = <u>467,000,000 cm</u>

(k) 2.58×10^0 cm = <u>25.8 mm</u>

(l) $.0057 \times 10^3$ m = <u>5700 mm</u>

2.8 Negative Exponents. Scientific Notation

The use of negative numbers as exponents here does not imply the necessity of developing the negative numbers in general at this time. The negative exponent is only an alternative notational form to the use of positive exponents in the denominators. It would be better to defer a discussion of negative numbers until the number line is developed.

The student should discover for himself that exponents add when powers are multiplied. He should also discover for himself the rules for the addition of signed numbers. In some cases the student will also discover that exponents subtract when powers are divided.

Exercise 7

Express in scientific notation. Use a World Almanac or similar reference book to get the appropriate numbers where they are needed.

1. The speed of light correct to two significant figures.

   3.0×10^8 m/sec or 1.9×10^5 mi/sec

2. The speed of light correct to five significant figures.

   2.9979×10^8 m/sec

3. The number of Angstrom units in one centimeter.

   1×10^{-8} cm

4. The number of meters in one millimeter.

   1×10^{-3} m

5. The number of kilometers in one millimeter.

   1×10^{-6} km

6. The annual budget of the United States.

   $\$ 9.6 \times 10^{10}$ (1964)

7. The magnitude of the area of the United States in square miles.

   U. S. A. area 3.615222×10^6 mi^2

8. The magnitude of the area of your state in square miles.

   Alaska area 5.864×10^5 mi^2; Connecticut area 5.009×10^3 mi^2

9. The ratio of the number in Problem 7 to the number in Problem 8. Carry out the indicated calculations and write the answer in scientific notation.

$$\frac{3.6 \times 10^6}{5 \times 10^3} = 7.02 \times 10^2 = \frac{\text{Area (U.S.A.)}}{\text{Area (Conn.)}}$$

Carry out the indicated calculations and write the answer in scientific notation.

10. $(1.6 \times 10^3) \times (1.1 \times 10^2) = \underline{1.76 \times 10^5}$

11. $(1.6 \times 10^3) \times (1.1 \times 10^{-2}) = \underline{1.76 \times 10}$

12. $(1.6 \times 10^{-3}) \times (1.1 \times 10^2) = \underline{1.76 \times 10^{-1}}$

13. $(1.6 \times 10^{-3}) \times (1.1 \times 10^{-2}) = \underline{1.76 \times 10^{-5}}$

14. Use the facts that 36 in = 1 yd and 2.54 cm = 1 in to express 1 cm in yards.

1.09×10^{-2} yd

15. If the speed of light is 1.86×10^5 miles per second, change it to meters per second. Use 1 mile = 5280×12 in and 1 in = 2.54 cm.

2.99×10^8 m/sec

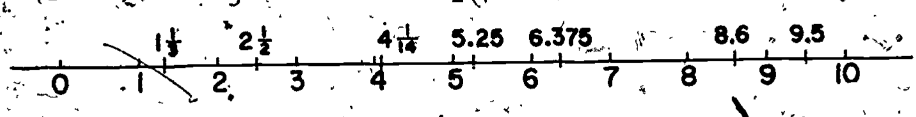
2.9 The Number Line

This section introduces the student to the number line. Only positive numbers are indicated to allow for the possibility that the instructor may not wish to discuss negative numbers at this class level. However, the zero is placed inside the line so that the instructor can raise the question of what numbers can be assigned to the points on the left of zero.

For further discussion see Mathematics for Junior High School, Volume 2, Part I, Chapter 6.

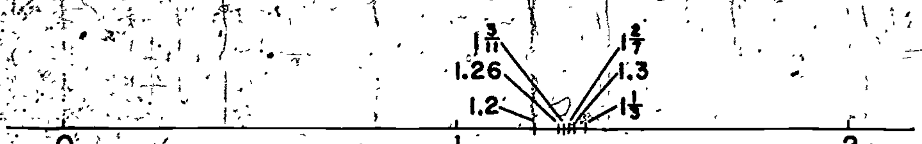
Exercise 8

1. Draw your own number line to cover the first ten counting numbers. Mark the following numbers on it as accurately as you can: $2\frac{1}{2}$, 9.5, 8.6, $1\frac{1}{3}$, 5.25, 6.375, $4\frac{1}{14}$.

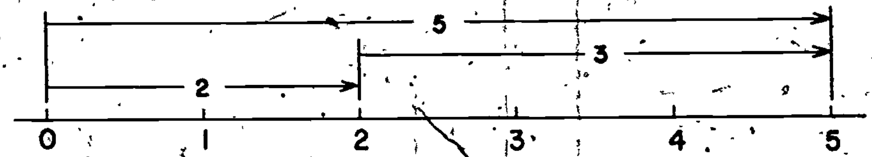


2. On another line, making each unit 10 cm in length, mark the following numbers: $1\frac{1}{3}$, 1.26, $1\frac{2}{7}$, 1.3, $1\frac{3}{11}$, 1.2.

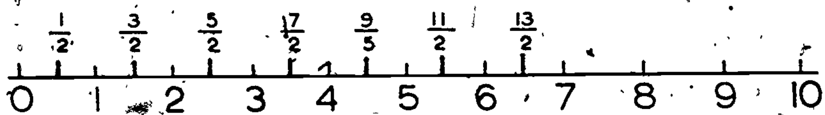
Arrange these numbers in ascending order of magnitude.



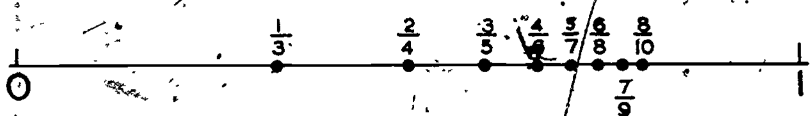
3. Illustrate $2 + 3 = 5$ on a number line.



4. On the number line mark the points representing the following numbers:
$\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \frac{9}{5}, \frac{11}{2}, \frac{13}{2}$.



5. (a) On the number line mark the points representing the following numbers: $\frac{1}{3}, \frac{2}{4}, \frac{3}{5}, \frac{4}{6}, \frac{5}{7}, \frac{6}{8}, \frac{8}{10}, \frac{7}{9}$.



(b) Which of these rational numbers in part (a) can be written as exact decimals?

$\frac{2}{4} = .5; \ \frac{3}{5} = .6; \ \frac{6}{8} = .75; \ \frac{8}{10} = .8$

6. If it is possible, express each of the following numbers as a simple fraction whose numerator and denominator are both integers.

(a) .25 $\qquad \frac{1}{4}$

(b) .333... $\qquad \frac{1}{3}$

(c) π $\qquad$ not possible

(d) .125 $\qquad \frac{1}{8}$

(e) $\sqrt{2}$ $\qquad$ not possible

(f) .8333 ... $\qquad \frac{5}{6}$

(g) .777 ... $\qquad \frac{7}{9}$

(h) $\sqrt{4}$ $\qquad 2$

(i) .363636 ... $\frac{4}{11}$ solution to (i):

$$x = .363636 \ldots$$
$$100x = 36.363636 \ldots$$
$$\text{subtracting } x = .363636 \ldots$$
$$99x = 36$$
$$x = \frac{36}{99} = \frac{4}{11}$$

(j) .454545 ... $\frac{5}{11}$

7. Express each of the following as an exact decimal wherever possible.

(a) $\frac{1}{8}$ .125

(b) $\frac{3}{7}$ not possible

(c) $\frac{9}{20}$ .45

(d) $\frac{7}{12}$ not possible

(e) $\frac{5}{11}$ not possible

(f) $\frac{5}{6}$ not possible

(g) $\frac{23}{16}$ 1.4375

(h) $\frac{2}{13}$ not possible

Sample Test Items

1. Fill in the following blanks:

   (a) 7 kilometers = ______________ meters.

   (b) 265 millimeters = ______________ kilometers.

   (c) 2178 centimeters = ______________ millimeters.

   (d) 635 centimeters = ______________ kilometers.

2. Write down the first five successive estimate inequalities, in decimal form, to the following length measures:

   (a) 94.1297473 (b) $4\frac{1}{4}$

   ______________ ______________

   ______________ ______________

   ______________ ______________

   ______________ ______________

   ______________ ______________

3. Write the inequality implied by the following measure statement:

   m is approximately 8.6 .

4. Write these numbers in a form that does not contain exponents.

   (a) 36.745×10^4 = ______________

   (b) 3.2×10^{-3} = ______________.

5. Write the following numbers in scientific notation:

   (a) 6,575,000 = ______________

   (b) .000893 = ______________

   (c) 3.47 = ______________

6. Carry out the indicated multiplications and write the answer in scientific notation.

   $(1.4 \times 10^4) \times (1.3 \times 10^{-2})$ = ______________

7. Draw a number line representing the first 10 counting numbers and locate the following points on it:

   $3\frac{1}{2}$, 7.25, $2\frac{1}{7}$, $1\frac{1}{3}$, $8\frac{1}{10}$

Answers to Sample Test Items

1. (a) 7000 m
   (b) .000265 km
   (c) 21780 mm
   (d) .00635 km

2. (a)

$$94 \le m < 95$$
$$94.1 \le m < 94.2$$
$$94.12 \le m < 94.13$$
$$94.129 \le m < 94.130$$
$$94.1297 \le m < 94.1298$$

   (b)

$$4 \le m < 5$$
$$4.2 \le m < 4.3$$
$$4.25 \le m < 4.26$$
$$4.250 \le m < 4.251$$
$$4.2500 \le m < 4.2501$$

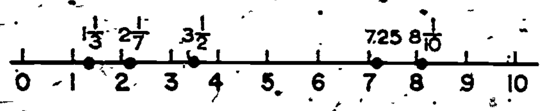
3. $8.55 < m < 8.65$

4. (a) 367,450
   (b) .0032

5. (a) 6.575×10^6
   (b) 8.93×10^{-4}
   (c) 3.47×10^0

6. 1.82×10^2

7.



Chapter 3

RELATIONS, FUNCTIONS AND GRAPHING

3.1) Introduction

In this chapter, three mathematical concepts are developed: (1) the definition of a relation, (2) the definition of a function, and (3) graphing order pairs by a coordinate system in a plane. In addition to these three concepts, it is hoped that the student will gain a strong feeling for the meaning of the domain and range of relations and functions.

This material is developed with the aid of two rather simple experiments: (1) the cantilever of books, and (2) the graduating of an irregular bottle. The first experiment deals with a function whose range and domain are discrete. The second experiment develops a function which is continuous. It is possible to consider fractions of glugs, and therefore, intermediate points can be found. Both experiments should point out to the student a need to consider very carefully all limitations placed upon the domain and range of a function or a relation.

A good reference for relations, functions, domain, range and mappings may be found in Principles of Mathematics, Allendoerfer and Oakley, McGraw-Hill, 1963, Chapter 6.

As in all experiments, students should use care in obtaining data from the experiments in this chapter. Accuracy in measuring, in reading results as well as in performing the experiments will lead to more satisfying results.

Students should be cautioned to keep their data since it will be used in subsequent sections and chapters. Since the data is to be retained, it would be well to put it in such a form that it will be intelligible at a later date (see below).

Domain (Name units)	Range (Name units)
______	______
______	______
______	______
______	______

Figure 1

3.2 Ordered Pairs

The emphasis in this section is on ordering pairs of things. The practice and development of ordering numbers and things, as expressed in this chapter, should make it seem very natural to the student. The following example is "very close" to graphing, and care must be used not to develop this point too far. Graphs will be carefully developed in subsequent sections.

The position of students in the classroom as expressed in row and seat numbers is a set of ordered pairs that can readily be used as an example. Row 2 seat 5 might be the seat that John occupies, Row 3 seat 4 the seat assigned to Jim, and row 5 seat 2 is Tom's seat. As ordered pairs these might be written as (2,5), (3,4) and (5,2). It must be noted that unless the importance of order is stressed, the positions of John and Tom could be confused. Order, then, is a necessity if consistent understanding and agreement is to be reached. Note, too, that this set of ordered pairs does not constitute a meaningful relation.

Supplementary material may be found in

"Mathematics for Junior High School", Volume 2, Part 1, SMSG;
"Exploring Modern Mathematics", Book 2, Keedy, Jameson and Johnson, Holt, Rinehart and Winston, 1963;

and many other books.

3.3 Relations

In the text, a relation is defined as a set of ordered pairs. This may seem strange. However, we note that a set M of ordered pairs defines a relation R in the following fashion. X R Y, (X is related to Y), is true if there is an element of M having X as its first member and Y as its second member; otherwise, X R Y is false. The set of first elements of the ordered pairs in M is the domain, and the set of second elements is the range. The relation pairs elements of the domain with those of the range.

For example, if R is the relation "less than" ($<$) and X is the set of real numbers, then $2 < 3$ is meaningful and true. More generally, this relation is the set of ordered pairs contained in the coordinate plane between the line $x = y$ and the positive x-axis.

Exercise 1

1. Make a set of ordered pairs by assigning to each month of a leap year the number of days it contains.

   (a) How many ordered pairs are in the set?

   (b) How many elements are in the domain of the relation?

   (c) Write the set of elements in the range of the relation. (Hint: In a set an element is listed only once.)

   (d) How many elements are in the range of the relation?

   (January, 31 days) (July, 31 days)
   (February, 29 days) (August, 31 days)
   (March, 31 days) (September, 30 days)
   (April, 30 days) (October, 31 days)
   (May, 31 days) (November, 30 days)
   (June, 30 days) (December, 31 days)

   (a) There are 12 ordered pairs in the set.

   (b) There are 12 elements in the domain of the relations. January, February, March, etc.

   (c) Set of elements in the range: {29, 30, 31}

   (d) There are 3 elements in the range of the relation.

2. Suppose we had used the year following a leap year in Problem 1. Would you have answered Problems 1(a - d) any differently? Why?

   Yes. 1(c) would have been answered differently, since the set of elements in the range would be {28, 30, 31}. February has 28 days except during a leap year.

3. The domain of a certain relation is the set {7, 10, 22, 1}; the range is the set {B, C, A, D}. From this information form one relation having five ordered pairs and one relation having seven ordered pairs.

   There are many relations that can be obtained from the information given in Problem 3. These answers are two of the many.

   {(7,B), (7,C), (10,C), (22,A), (1,D)}

   {(7,B), (10,B), (7,A), (10,A), (7,C), (10,C), (7,D)}

4. Use the index in the back of one of your texts to obtain ordered pairs.

   (a) List four ordered pairs where different elements of the domain share the same element of the range.

   (b) List four ordered pairs where different elements of the range are assigned to the same element of the domain.

The SMSG text Mathematics for Junior High School, Volume 2, Part II, was used to illustrate a possible solution to this exercise.

(a) (additive inverse, 237), (multiplicative inverse, 237) (repeating decimal, 247), (terminating decimal, 247)

(b) (exponent, 121), (exponent, 126), (exponent, 129), (exponent, 149)

5. Given the following sets of ordered pairs:

(a) {1, 2}, {4, 2}, {5, 2}, {6, 3}, {7, 3}
(b) {0, 0}, {1, 0}, {2, 0}, {3, 0}

determine the domain and range.

(a) Domain {1, 4, 5, 6, 7}. Range {2, 3}
(b) Domain {0, 1, 2, 3}. Range {0}

3.4 An Experiment (Cantilever experiment)

The cantilever experiment requires the following materials for each group of students.

5 identical books
or
5 identical boards
1 foot ruler, with metric scale

If books are used, they should have hard covers. Paper covered or soft books will bend and, therefore, are not suitable for this experiment.

Although balancing the books is not too difficult, the student may experience some trouble making an accurate measurement of the overhang. It may be much simpler for him to measure to the nearest centimeter. The ruler can be placed with its end against the edge of the table under the bottom book. A second ruler, or plumb, can be hung close to the top book to give its projection on the measuring ruler. This is illustrated in Figure 2.

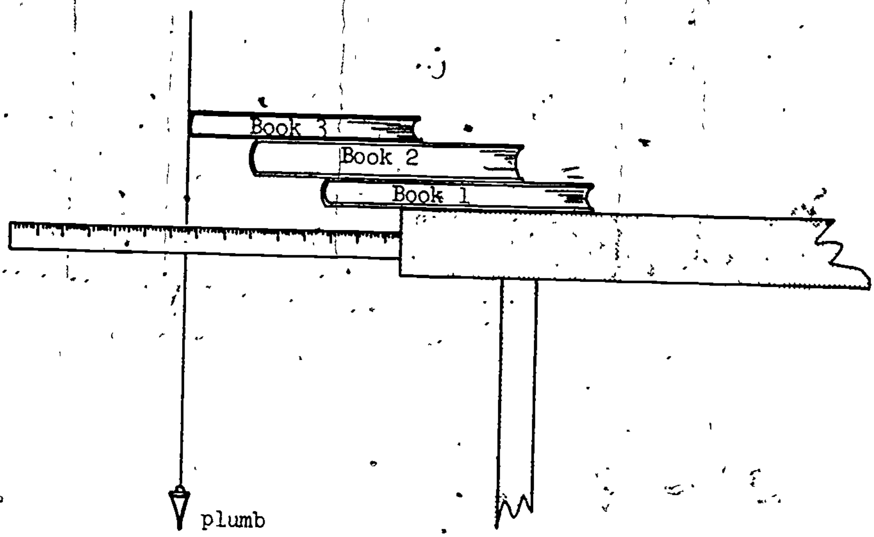


Figure 2

Emphasize to the student that the numbers used in the text are only examples. The numbers the student obtains will differ from the example given for the data depends on many factors.

Finally, it should be noted that the second numbers of the ordered pairs are not obvious, and prejudging the results will usually lead to error.

3.5 Graphing of Ordered Pairs

Before launching into this section, it would be well to determine student understanding of coordinates and of the real numbers.

Most students will already know that a coordinate is a point on the number line associated with a particular value. He must also recognize that the number must tell both the distance and direction of the point from the 0-point if it is to completely fulfill the conditions necessary for a coordinate.

The set of all numbers which are associated with points on the number line may be called the set of real numbers. Here the direction from the 0-point differentiates between the negative real numbers and the positive real numbers.

For further discussion of the topics, refer to

"Mathematics for the Elementary School", Grade 6 Parts 1 and II, SMSG;

"First Course in Algebra", Parts I and II, SMSG;

"Mathematics For Junior High School", Volume 2, Part I, SMSG.

Exercise 2

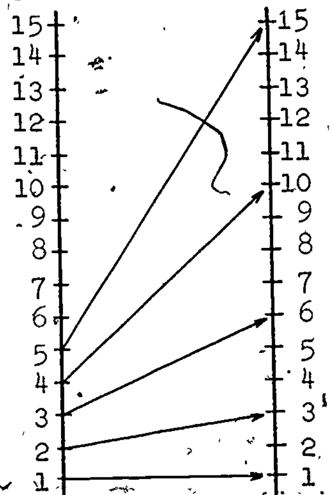
Graph (by drawing arrows from one number line to another) each of the following sets of ordered pairs.

1. {(1,1), (2,3), (3,6), (4,10), (5,15)}

   (a) What are the elements of the domain?

   (b) What are the elements of the range?

   (c) If a relation contained the above ordered pairs, and if the number 6 were also an element in the domain of this relation, what would you guess for the corresponding element in the range if the same pattern were repeated?



(a) domain: {1,2,3,4,5}

(b) range: {1,3,6,10,15}

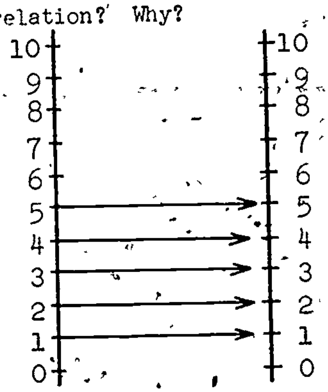
(c) 21 could be a good guess since the graph shows that the difference of the elements in the range for succeeding order pairs increases by one as the element in the domain increases by a value of one.

2. {(1,1), (2,2), (3,3), (4,4), (5,5)}

   (a) What are the elements of the domain?

   (b) What are the elements of the range?

   (c) Is the ordered pair (6,6) also an element in the set defined by this relation? Why?



(a) domain: {1,2,3,4,5}

(b) range: {1,2,3,4,5}

(c) No. The element 6 is not contained in either the set of elements of the domain or the set of elements of the range.

3. {(1,3), (2,2), (3,3), (4,2), (5,3), (6,2)}

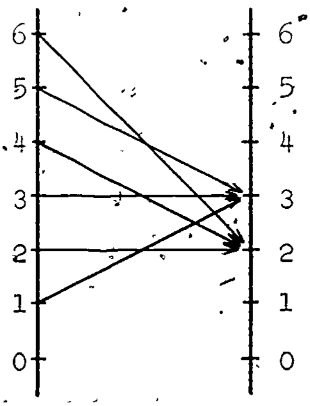
(a) What are the elements of the domain?

(b) What are the elements of the range?

(c) If a relation contained the above ordered pairs and if the number 15 were also an element in the domain of this relation, what would you guess for the corresponding element in the range if the same pattern were continued?

(d) If the number 8 were an element in the domain of a relation similar to that described in part (c), what would you guess for the corresponding element in the range if the same pattern were continued?

(e) State a rule for writing ordered pairs belonging to a relation similar to parts (c) and (d) if the domain were the counting numbers 1 through 100.



(a) domain: {1,2,3,4,5,6}

(b) range: {2,3}

(c) 3

(d) 2

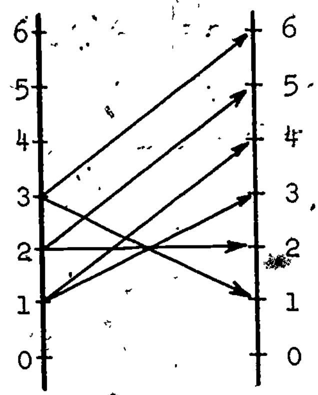
(e) If the element of the domain is an even number, then the element of the range assigned to it is the number 2. If the element of the domain is an odd number, then the element of the range assigned to it is the number 3.

4. {(3,1), (2,2), (1,3), (1,4), (2,5), (3,6)}

(a) What are the elements of the domain?

(b) What are the elements of the range?

(c) If a relation contained the above ordered pairs, and if the number 4 were also in the domain of this relation, what would you guess for the corresponding element in the range if the same pattern were continued?



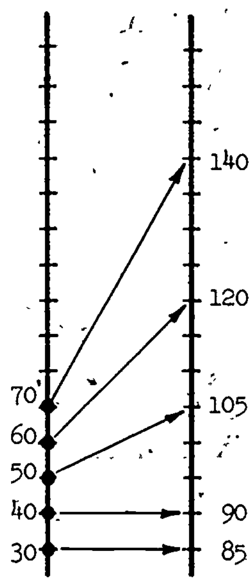
(a) domain: $\{1,2,3\}$.

(b) range: $\{1,2,3,4,5,6\}$

(c) 0 or 7

Each of the following graphs, (5 - 10), defines a relation. List a set of ordered pairs which defines the same relation. List the domain and the range for each relation.

5.



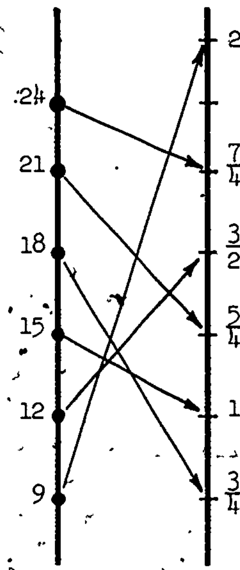
The ordered pairs are:

(30,85), (40,90), (50,105), (60,120), (70,140)

domain: $\{30,40,50,60,70\}$

range: $\{85,90,105,120,140\}$

6.



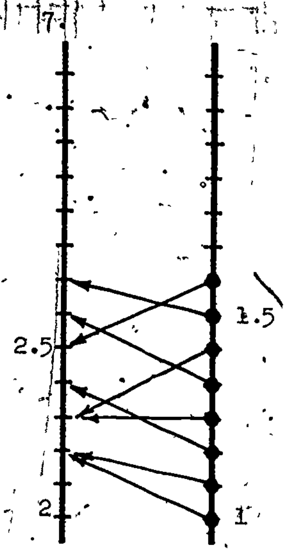
The ordered pairs are:

$(9,2)$, $(12,\frac{3}{2})$, $(15,1)$, $(18,\frac{3}{4})$, $(21,\frac{5}{4})$, $(24,\frac{7}{8})$

domain: $\{9,12,15,18,21,24\}$.

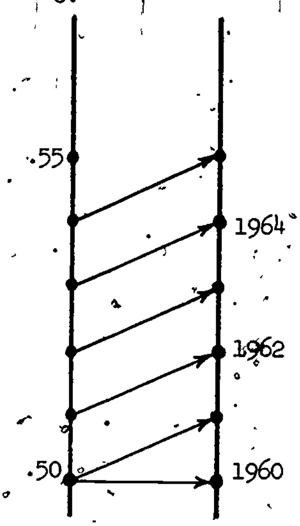
range: $\{2,\frac{3}{2},1,\frac{3}{4},\frac{5}{4},\frac{7}{8}\}$

7.



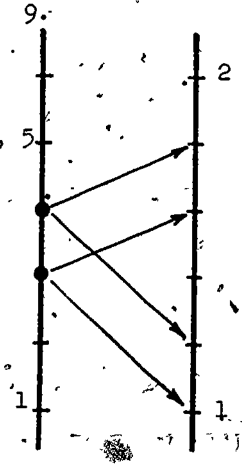
The ordered pairs are:
(1,2.2), (1.1, 2.2), (1.2,2.4),
(1.3,2.3), (1.4,2.6), (1.5,2.3),
(1.6,2.7), (1.7,2.5)
domain: {1,1.1,1.2,1.3,1.4,1.5,1.6,
1.7}
range: {2.2,2.3,2.4,2.5,2.6,2.7}

8.



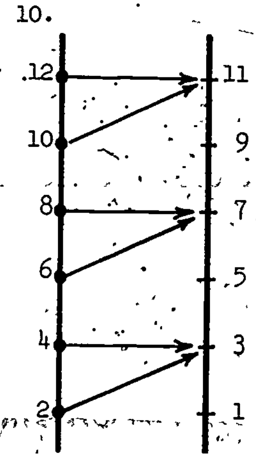
The ordered pairs are:
(.50,1960), (.50,1961), (.51,1962)
(.52,1963), (.53,1964), (.54,1965)
domain: {.50,.51,.52,.53,.54}
range: {1960,1961,1962,1963,1964,
1965}

9.



The ordered pairs are:
(3,1), (3,1.6), (4,1.2), (4,1.8)
domain: {3,4}
range: {1,1.2,1.6,1.8}

10.



The ordered pairs are:
(2,3), (4,3), (6,7), (8,7), (10,11)
(12,11)
domain: {2,4,6,8,10,12}
range: {3,7,11}

3.6 Functions

This section treats one of the most important and basic ideas in mathematics, -- the idea of a function. Previous sections have laid the groundwork. The idea of a function is simple, and should be readily accepted by the student. The concept of a function will be used extensively in the forthcoming chapter.

An extension of the definition of function is the idea of associating elements from one set with those of another set.

> Given a set of numbers and a rule which assigns to each number of this set exactly one number, the resulting association of numbers is called a <u>function</u>. The given set is called the <u>domain of definition</u> of the function, and the set of assigned numbers is called the <u>range</u> of the function.

The definition of function which we have given is, strictly speaking, a definition of <u>real function</u> since we have restricted the domain and range to real numbers. In later courses, the student will meet more general types of functions in which the domain and range can be sets other than sets of real numbers. Such a function might, for example, have sets of points in the plane as its domain of definition.

In the discussion about functions, it is important to emphasize at every opportunity the following points:

(1) To each number in the domain of definition, the function assigns <u>one and only one</u> number from the range. In other words, we do not have "multiple-valued" functions. However, the same number can be assigned to many different elements of the domain.

(2) The essential idea of function is found in the actual association from numbers in the domain to numbers in the range and <u>not</u> in the particular way in which the association happens to be described.

(3) Always speak of the association as being <u>from</u> the domain <u>to</u> the range. This helps fix the correct idea that we are dealing with an ordered pairing of numbers in which the number from the domain is mentioned first and the assigned number from the range is mentioned second.

(4) Not all functions can be represented by algebraic expressions.

Although the above points are not absolutely vital as far as elementary work with functions is concerned, they become of central importance later. Also, many of the difficulties which students have with the idea of function can be traced to confusion on these matters. Therefore, it becomes important to make certain that the student understands these points from his very first contact with the function concept.

Exercise 3

Define a function which is suggested by the phrases in Problems 1 - 7. Be sure to specify the domain and the range of each function.

1. {(1,2), (2,3), (3,4), (4,5), (5,6), ...}

   (Hint: The symbol, ..., used in this manner, means that the ordered pairs continue indefinitely according to the pattern suggested.)

   Set of all counting numbers is the domain.
   Set of all counting numbers except 1 is the range.

2. {(1,1), (2,4), (3,9), (4,16), (5,25), ...}

   Set of all counting numbers is the domain.
   Set of all squares of the counting numbers is the range.

3. Election returns

   Set of all candidates is the domain.
   Set of all votes cast is the range.

4. Area of triangles

   Set of all triangles is the domain.
   Set of all areas is the range.

5. People's first names

   Set of all people is the domain.
   Set of all first names is the range.

6. With each positive integer associate its remainder after division by 5.
   (Example: 1 ÷ 5 = 0 + remainder 1, hence, (1,1))

   Set of all positive integers is the domain.
   range: {1,2,3,4,0}

7. Associate with each length of the diameter of a circle the length of the circumference of the corresponding circle.

Set of all positive numbers is the domain.

Set of all positive numbers is the range.

8. The cost of mailing a letter is determined from the weight of the letter as follows: it is 5¢ per ounce plus 5¢ for any fraction of an ounce.

Does this describe a function? Why?

What is the domain?

(Note that the Post Office will not accept a first class package which weighs more than 20 pounds.)

What is the range?

Complete the following ordered pairs in this relation: (domain is given in ounces)

(3.7,______), (5,______), (19.2,______)

A function is described because <u>one</u> and only <u>one</u> element of the range is assigned to <u>each</u> element of the domain.

The set of all positive numbers less than or equal to 20 is the domain.

The set of all positive multiples of 5 is the range.

(3.7,20), (5,25), (19.2,100)

3.7 More Graphing (The Irregular Bottle Experiment)

The irregular bottle experiment requires the following material for each group of students.

1 irregular shaped bottle
1 plastic pill bottle
1 roll Magic mending tape
1 foot ruler with metric scale
1 #10 can or equivalent

The experiment to be performed in this section provides an opportunity for the student to discover that there are different kinds of data to be obtained from experiments. In the cantilever of books experiment, sets of ordered pairs were obtained. The values were discrete. No part books were used, nor can be inferred from the data obtained. This is not true of the second experiment in this section. Parts of "glugs" could either be used in the experiment or inferred from the resulting data. To make this difference more meaningful to the student, both experiments must be completed.

At this point the student first encounters the idea of a continuous function. If students intuitively discover the idea of physical continuity and desire to pursue the matter, further discussion would not be inappropriate. It would be necessary to have the student find other sets of ordered pairs and then group them as to continuous or discrete relations. The Wick Experiment, which is described below, provides another example of a continuous function. However, physical continuity need not be discussed at this point. It may well be introduced at a later time.

The equipment for this second experiment should be kept as simple as possible. Two bottles are needed. One should be a salad dressing, vegetable oil, coke, or other irregularly shaped bottle. The contour of this bottle is very significant in that it determines the kind of a curve to be graphed in a later section of the text from the data obtained in this experiment. The more complex bottle contour will provide the more interesting graph. The second bottle, a small bottle to be used as the "glug", is often readily available in the student's home. Also, a small plastic bottle with a volume of about 22 cc can often be obtained from the local druggist. Scotch magic mending tape or a masking tape provide good tapes to be used. Students should be cautioned to mark the tapes carefully.

THE WICK EXPERIMENT:

The following experiment can be used as an alternate to the irregular bottle experiment in this section, or as a supplementary experiment which can be done by students who have extra time. It is also simple enough that the students can do it at home. The results are usually quite good and make an excellent graph.

Materials required:

paper towel	paper clip
ruler	clock or watch with a second hand
cup, glass, or jar	
Scotch tape	

Description of the experiment:

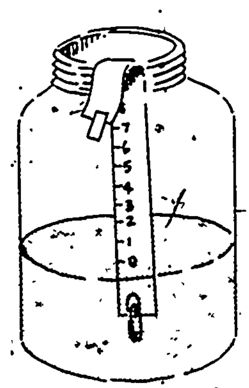
From a paper towel cut out a strip one inch wide by about eight to ten inches long. Draw a pencil line across the strip two or three centimeters from one end. Draw pencil lines one centimeter apart starting from this line. About ten lines will be sufficient. These lines should be numbered (also in pencil) with the first line starting at zero. Some care is needed in drawing and writing on the paper towel since it will tear quite easily.

Put some water in the jar. Fasten a paper clip to the lower end of the paper strip in the jar so that the "zero" line is exactly even with the surface of the water. Fasten it at this height with the Scotch tape. On the record sheet, note down the time this is done opposite the number zero. It is most convenient to do this exactly at an even minute to simplify later computations.

Record the time that the water rises to each line. The water level is seen most easily if the jar is placed against a dark background with the light coming from behind the viewer. The time should be recorded to the nearest quarter of a minute. The water usually does not rise exactly evenly and there is often some uncertainty about the exact time when the water reaches a given line. One way of judging this is to record the time of the quarter minute immediately before the first detectable water appears above the line (for this reason, it would be best to place the numbers below the lines).

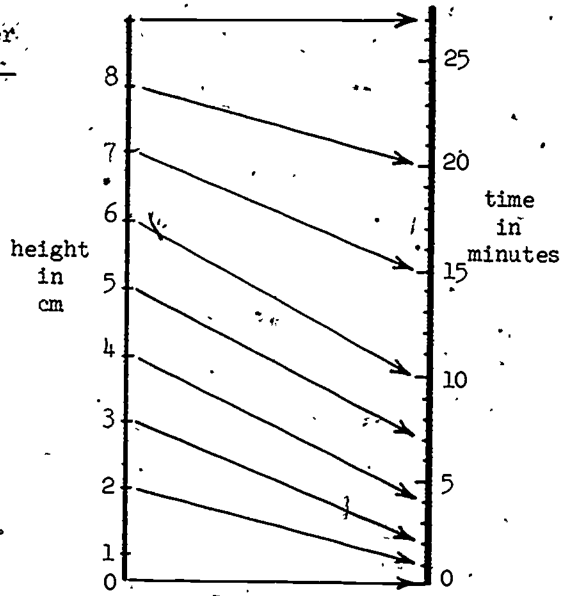
The wicking rate of paper towels vary, but the roll type, commonly used in home kitchens, is such that the water will rise about nine centimeters in half an hour. If this experiment is done in class, it would be wise to try it beforehand to determine how long the available paper towels would take. Seven or eight values will suffice for this experiment. If class periods are short, the paper strips can be prepared beforehand so that the experiment proper can start near the beginning of the period.

After all the data has been recorded, another column should be computed, showing the time from the start of the experiment required for the water to rise to the given level. A graph should then be prepared from this data.

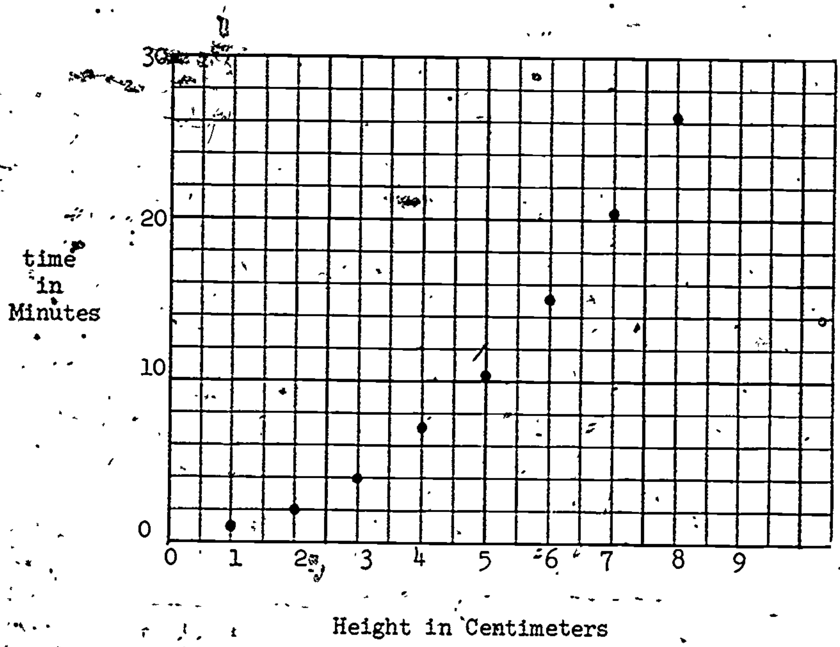


In the chart given below, we show typical data which might be obtained, and also the graph which would be prepared from this data.

Line	Time	Time after start
0	9:20:00	0:00
1	9:20:45	0:45
2	9:22:00	2:00
3	9:24:00	4:00
4	9:26:45	6:45
5	9:30:30	10:30
6	9:35:00	15:00
7	9:40:15	20:15
8	9:46:30	26:30



In the next section we will introduce rectangular coordinates for graphing. This same data graphed on rectangular coordinates looks like this.



If you get student questions, you should know that this is a quadratic curve. The student might find it interesting to graph the time required for the water to rise through each individual interval. This gives a linear graph.

3.8 A coordinate system in a plane

The rectangular coordinate system is introduced at this point since graphs are used extensively in the next chapter to display relations.

There are two ideas used in this section which may be unfamiliar to the student: These are perpendicular lines and negative numbers. There should be little trouble for the student if perpendicular lines are defined as any two lines which form a right angle.

Negative numbers will take a little more explanation. Only a very short explanation should be necessary, however, since not many operations involving negative numbers will be encountered. It is natural to use an extension of the number line to provide geometric reality and an intuitive feel for negative numbers. The idea of distance (or of points) along a line on opposite sides of a fixed point occurs frequently and can be used to fix these ideas. This has already been discussed in Section 4.5 in relation to coordinates and real numbers. A number line representing temperature provides an excellent example. Altitudes above and below sea level, location south and north of a given point are other possible examples. The order of rational numbers on the number line should be introduced. It is valuable to develop a natural acceptance of the fact that "is less than" means the same as "precedes on the number line". To say that one number "is greater than" another means simply that the one "follows" the other on the number line.

You will notice that in the discussion of the coordinate system in a plane care was taken not to overemphasize the x-axis or the y-axis. It was desired to give the student the feeling that the label attached to the horizontal axis and to the vertical axis would be dependent upon the sets which represent the domain and the range of the relation being graphed.

The fact that the horizontal axis is used to represent the domain is merely a convention and should be recognized as such. In some cases it may even be desirable to interchange the axis of the domain with the axis of the range. For example, if air temperature is to be plotted as a function of height, a stronger feeling for the physical situation may be gained by plotting height along the vertical axis instead of the horizontal axis.

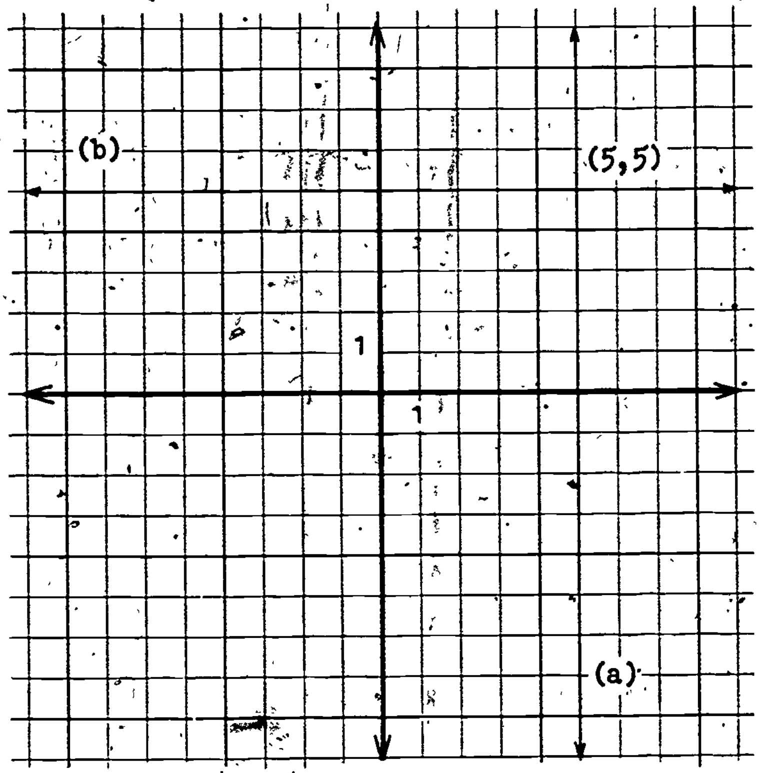
It would be most desirable to keep the vocabulary as simple as possible. Such words as abscissa, ordinate and variable have not been used for this reason. Also, the word variable introduces a generally obscure feeling for its definition. Bertrand Russell, who probably investigated the aspects of variables more thoroughly than anyone before him, said: "Variable is perhaps the most distinctly mathematical of all notions, it is certainly also one of the most difficult to understand . . . and in the present work (The Principles of Mathematics, 1903) a satisfactory theory as to its nature, in spite of much discussion, will hardly be found."

In classrooms where coordinate boards are not available, it is possible to make a semi-permanent coordinate board. Soak hard white chalk in a solution of sugar water for about 20 minutes. Using the wet chalk, carefully draw the axes and as many other lines as are deemed necessary. Wet the chalk often as the lines are being drawn. When dry this should provide lines that will not easily erase. When desired, they can be removed with a damp rag.

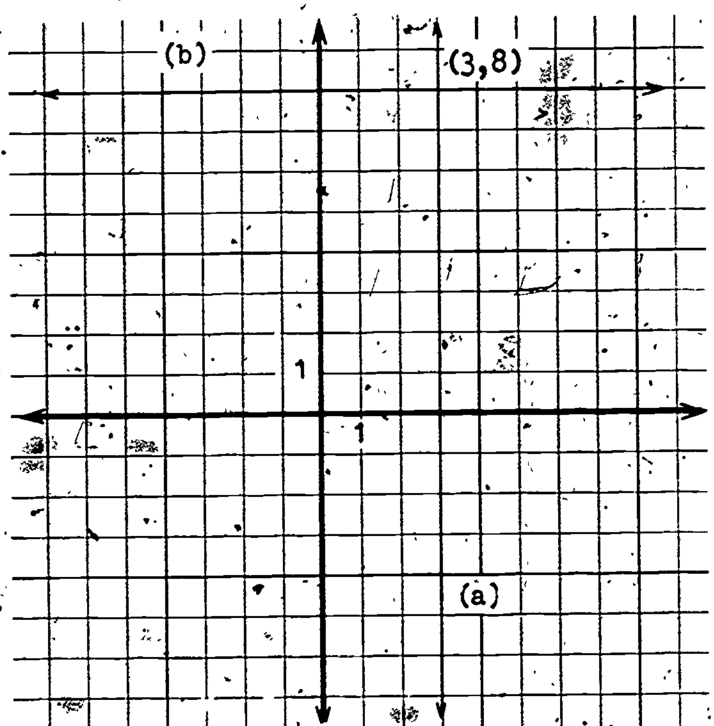
Ordered pairs can be plotted on this board with regular chalk. Lines and points can be drawn on this board and erased from it without disturbing the original lines.

Exercise 4

1. On squared graph paper draw a pair of axes and label them. Indicate the coordinate system on each axis.
   - (a) Sketch (with a straight edge) a line which represents the set of points whose horizontal coordinate is 5.
   - (b) On the same coordinate plane sketch a line which represents the set of points whose vertical coordinate is 5.
   - (c) How many points do these two sets have in common?
   - (d) Write as an ordered pair the coordinates of every point of intersection of the sets graphed in (a) and (b).



2. Repeat Problem 1 for the set of points whose horizontal coordinate is 3 and the set of points whose vertical coordinate is 8.



(c) The two sets have <u>one</u> <u>point</u> in common.

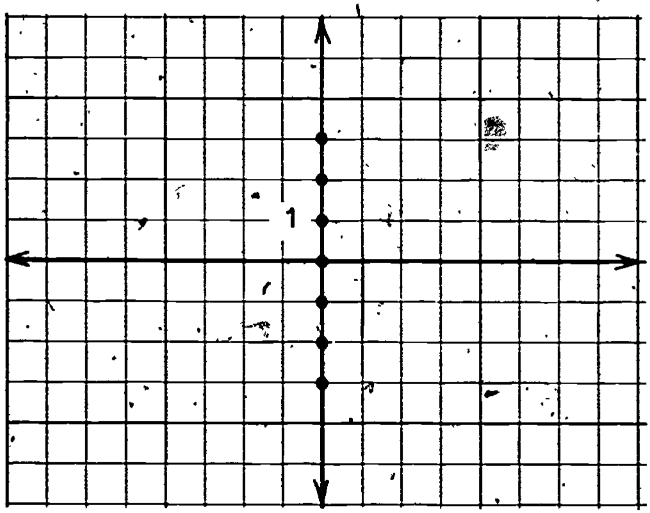
(d) (3,8)

(b) Do all the points in this set seem to lie on the same line? Yes.

(c) What do you notice about the vertical coordinate for each of the points? The vertical coordinate for each point is <u>0</u>.

6. (a) Plot the points in the following set:

$\{(0,0), (0,-1), (0,1), (0,-2), (0,2), (0,-3), (0,3)\}$



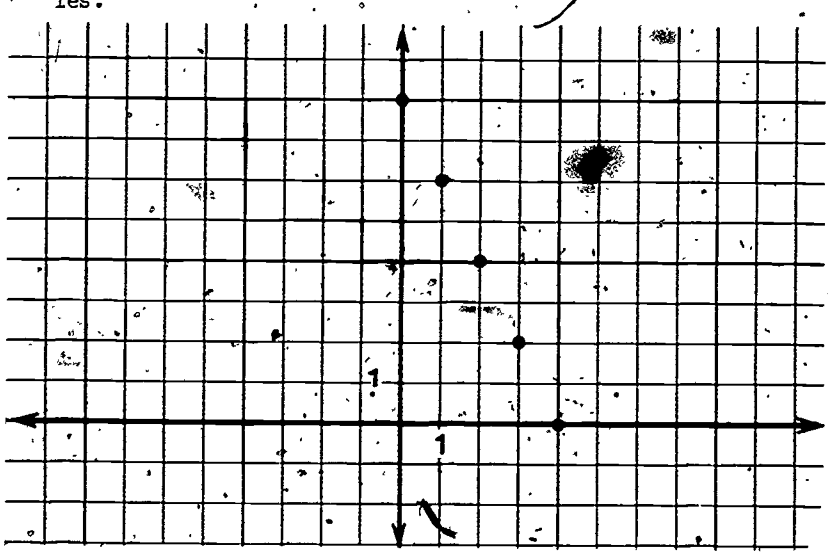
(b) Do all the points named in this set seem to be on the same line? Yes.

(c) What do you notice about the horizontal coordinate for each of the points? Each point has the horizontal coordinate <u>0</u>.

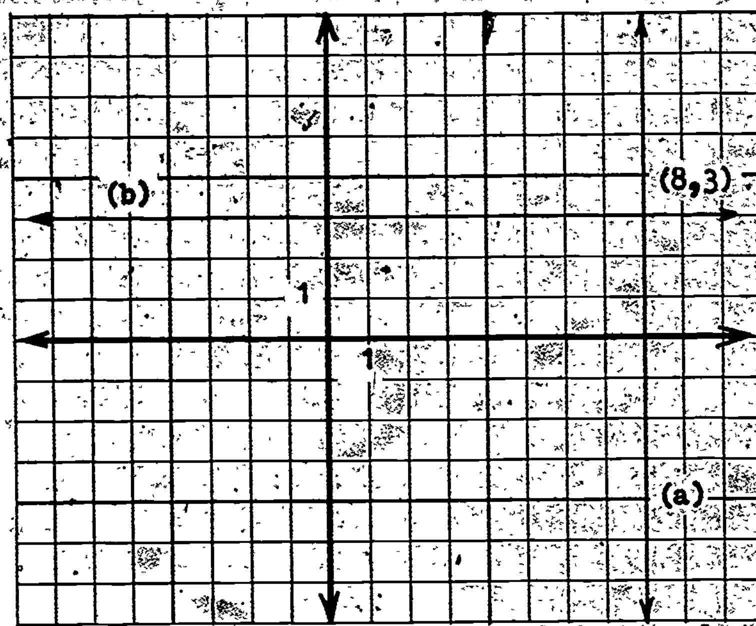
7. (a) Plot the points in the following set:

$\{(0,8), (1,6), (2,4), (3,2), (4,0)\}$

(b) Do all the points named in this set seem to lie on the same line? Yes.



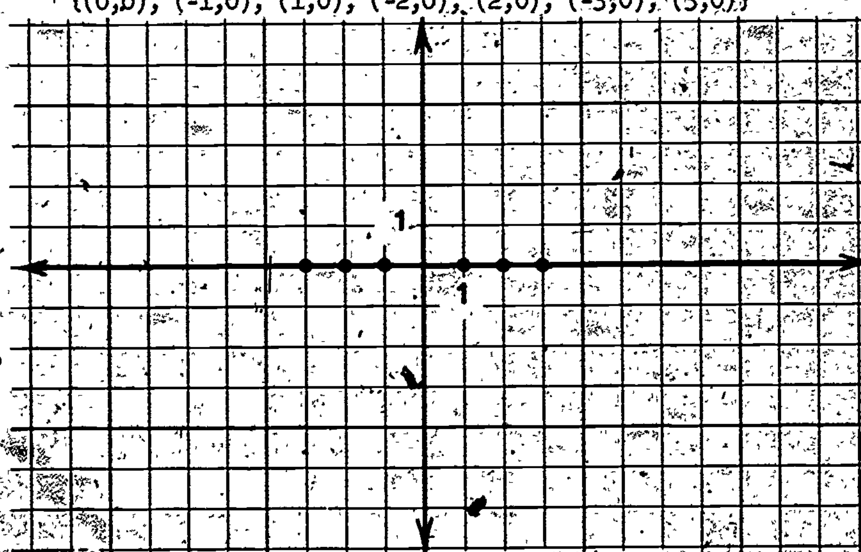
3. Repeat Problem 1 for the set of points whose horizontal coordinate is 8 and whose vertical coordinate is 3.



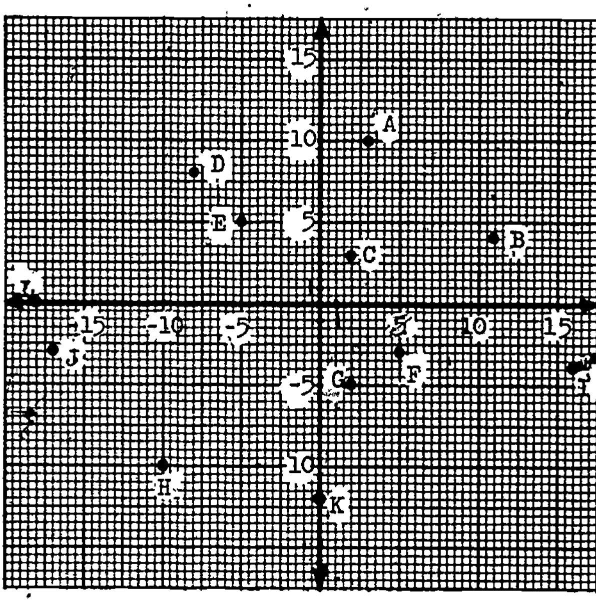
4. Is the point of intersection of the two sets in Problem 2 the same point as the point of intersection of the two sets in Problem 3? Why?

No. The domain in Exercise 2 is 3, while the domain in Exercise 3 is 8. The range in Exercise 2 is 8, while the range in Exercise 3 is 3.

5. (a) Plot on a coordinate plane the following set of points: $\{(0,0), (-1,0), (1,0), (-2,0), (2,0), (-3,0), (3,0)\}$



8. Write the coordinates of each point in the following graph as an ordered pair:



A(3,10) B(11,4) C(2,3) D(-8,8) E(-5,5) F(5,-3) G(2,-5)
H(-10,-10) I(16,-4) J(-17,-3) K(0,-12) L(-18,0)

3.9 Quadrants

It is specified in the text that points with one or both elements zero lie on the axes. Points on either the x-axis or the y-axis do not lie in a definite quadrant. Therefore, the quadrants do not contain the axes or the origin.

Exercise 5

1. Given the following ordered pairs of numbers, write the number of the quadrant or the position on an axis in which you find the point represented by each of these ordered pairs:

(a) (3,5) I
(b) (-5,1) II
(c) (1,-4) IV
(d) (-4,4) II
(e) (0,0) origin
(f) (0,5) positive vertical axis
(g) (-3,-1) III
(h) (7,-1) IV
(i) (8,6) I
(j) (3,-2) IV
(k) (-3,-5) III
(l) (-1,3) II
(m) (2,-4) IV
(n) (5,2) I
(o) (-3,0) negative horizontal axis
(p) (-4,-5) III
(q) (-1,2) II
(r) (3,-1) IV

3.10 Graphing an Experiment

There are a number of items that should be emphasized in the graphing of these experimental points. Since the data contains no negative numbers, it is necessary to display only the first quadrant. Students should decide on a scale which will graph the ordered pairs over most of the page. Further, it would be most convenient if everyone uses this same scale. The large scale is especially important when it is necessary to draw the "best straight line", or, later, when it is necessary to draw the "best curve". If a small scale is chosen, the points will be confined to a small area of the page. In a small area it is difficult to draw a "best" straight line.

If the experiment has been performed carefully and somewhat accurate data obtained, it is not difficult to fit a good line (or curve) to the data. If some points are obviously out of place, perhaps due to incorrectly reading a scale, the student should be advised to ignore this point in fitting the curve.

The data from the two experiments with which we have been concerned in this chapter express relations which differ from one another in fundamentals. In the cantilever of books it makes no physical sense to cut the books into part books. There are no values of the domain between the integers and hence the five ordered pairs are a complete set. In addition, it makes no physical sense to extend the domain to larger values, for there is a maximum. Nor is there any meaning to negative values in this data. The student must recognize that connecting the points or extending the graph is not meaningful.

On the other hand, in the irregular bottle experiment it is meaningful to talk of parts of glugs or more glugs. If the volume of the bottle is greater, more glugs can be added. There is, however, a maximum. As in the cantilever of books experiment, negative glugs have no meaning, and there can be no extension to other quadrants.

Although we examined the domain and range of the relation on physical grounds, it is necessary to be aware of them in nonphysical cases. The student must be made to realize that the relation or function carries with it a statement of the domain and range over which the relation is valid.

Exercise 6

1. In an experiment a solid material was heated over a burner, and the temperature was recorded as a function of the time it took the material to reach a definite temperature. Let the domain be the set of all times

from the beginning of the experiment to 18 minutes later and the range be the set of all temperatures from 20°C to 200°C.

(a) Draw a horizontal and vertical axis and place the appropriate labels for the domain and range on these axes.

(b) Mark the domain on the horizontal axis with a heavy dark line as in Figure 21.

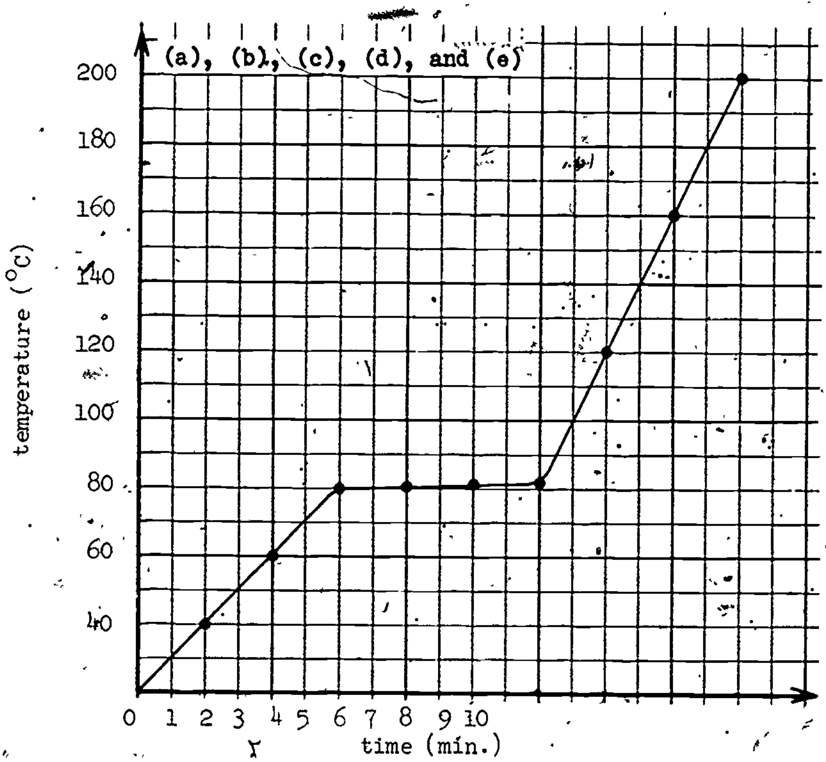
(c) Mark the range on the vertical axis.

(d) Plot the following ordered pairs which the "scientist" collected when doing the experiment: (0,20), (2,40), (4,60), (6,80), (8,81), (10,82), (12,83), (14,120), (16,160), (18,200).

(e) Connect these points with a "smooth" curve.

(f) What temperature would you predict for the material ~~at the~~ following times: 1 minute, 5 minutes, 9 minutes, 17 minutes?

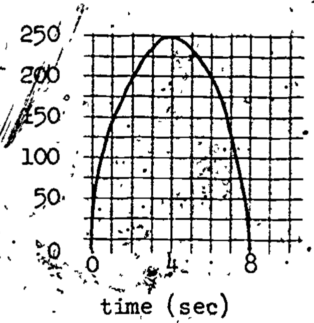
(g) At what time would you predict the material would have the following temperatures: 70°C, 100°C, 150°C, 180°C?



(f) 30°C at 1 minute, 70°C at 5 minutes, 81.5°C at 9 minutes, $82\frac{1}{2}$°C at 11 minutes

(g) 50°C at 3 minutes, 100°C at 13 minutes, 150°C at $15\frac{1}{2}$ minutes, 180°C at 17 minutes.

2. The following graph was drawn from information gathered in an experiment dealing with a ball thrown into the air. The height of the ball above the ground was plotted as a function of the time it took the ball to reach a definite height.



(a) What is the domain of this function?

Set of times in seconds

(b) What is the range of this function?

Set of heights in feet

(c) How high would you predict the ball would be after $\frac{1}{2}$ second?

44 feet

(d) How long had the ball been in upward flight when it reached a height of 128 ft?

2 seconds

(e) How long had the ball been in flight when it descended to a height of 128 ft?

4 seconds

(f) Can anything meaningful be said concerning the height of the ball after 10 seconds? Explain.

No, since the ball fell to earth at 6 seconds.

Sample Test Items

1. The following is a list of certain United States' Presidents and their ages at the time of inauguration. From this list make a set of ordered pairs.

George Washington 57 years; John Adams 61 years; Thomas Jefferson 57 years; William Taft 51 years; Woodrow Wilson 56 years; Franklin Roosevelt 51 years; Harry Truman 60 years.

(a) Ordered Pairs

(b) How many ordered pairs are in the set? ____________

(c) How many elements are in the domain? ____________

(d) How many elements are in the range? ____________

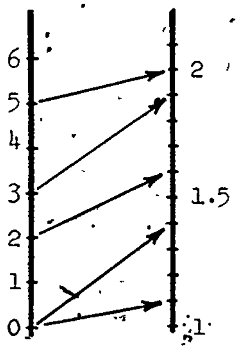
(e) What is the name given to a set of ordered pairs? ____________

2. Find the missing elements in the domain and range in the following set of ordered pairs:

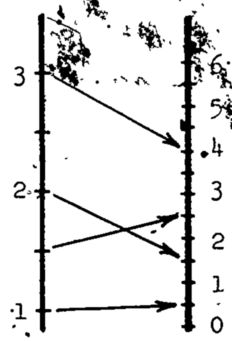
$\{(3,1), (9,3), (12,4), (\_\_,6)\ (\_\_,\frac{1}{2}), (1,\_\_), (\frac{2}{3},\_\_)\}$.

3. For the following graphs, write the set of ordered pairs which define the relation

(a)



(b)



4. Which of the following sets of ordered pairs are functions?

(a) {(2,3), (5,6), (0,1), (3,4), (4,5), (6,7)}

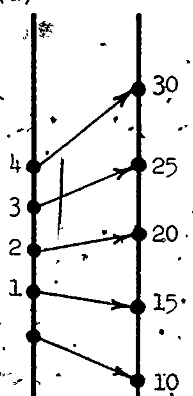
(b) {(3,2), (2,2), (9,4), (7,4), (4,9), (5,2)}

(c) {(2,1), (3,4), (5,1), (2,4), (5,2), (4,4)}

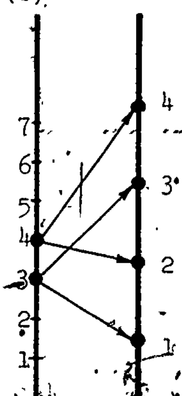
(d) {(2.5,2), (7.5,3), (1.5,2), (5.5,3), (3.5,2)}

5. Which of the following graphs represent a function?

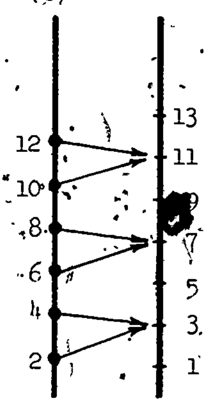
(a)



(b)



(c)



6. Plot the following sets of ordered pairs on the coordinate plane.

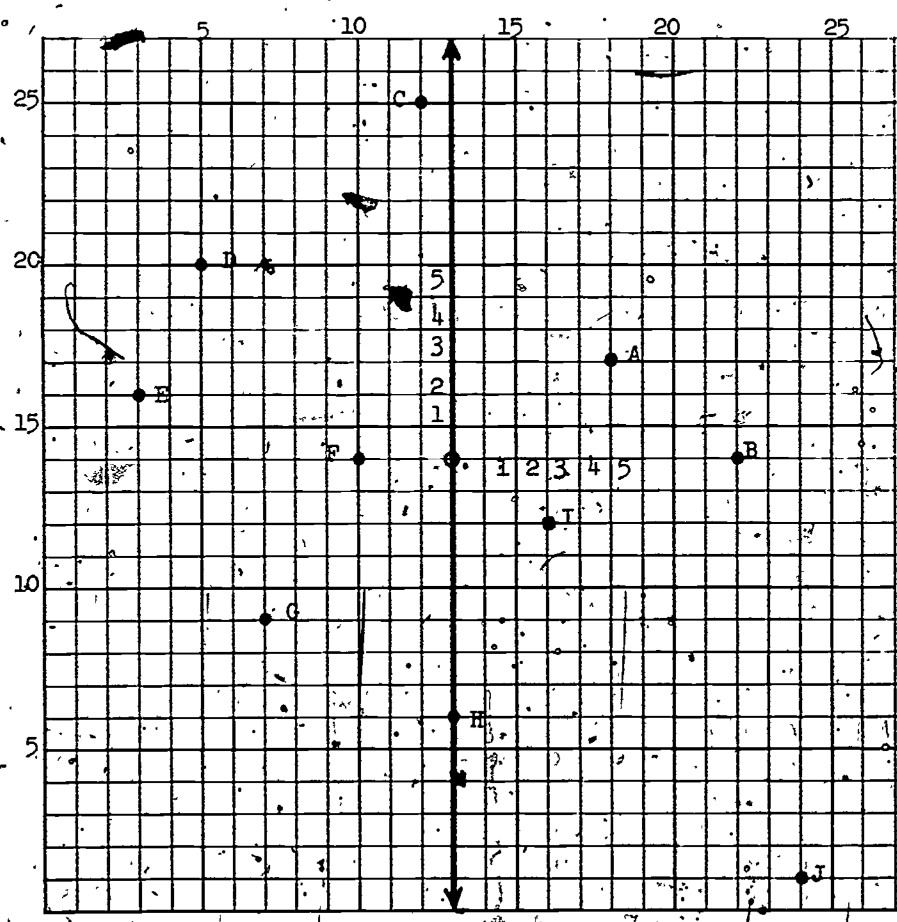
(a) (2,5) (e) (6,0)
(b) (3,-1) (f) (-3,5)
(c) (-4,9) (g) (0,-3)
(d) $(5\frac{1}{2},8)$ (h) (-2,-6)

7. Name the quadrant or the position on an axis in which is found the point represented by each of the following ordered pairs:

(a) (5,7) (d) (-3,-8) (g) $(5,2\frac{1}{2})$
(b) (-4,-2) (e) (-6,5) (h) (-2,0)
(c) (2,-4) (f) (-4,3) (i) $(0,3\frac{1}{4})$

8. Write the coordinates of each point in the following graph as an ordered pair.

A. D. G. J.
B. E. H.
C. F. I.



9. As part of a weather unit in science a student kept a record of noonday temperatures for the first ten days in December. The record follows. Draw a horizontal and a vertical axis and place the appropriate labels for the domain and range on these axes. Then plot the information recorded by the student as ordered pairs.

December	1	2	3	4	5	6	7	8	9	10
Temperature	38°F	45°	48°	32°	30°	42°	55°	36°	45°	40°

10. Several identical experimental rockets were fired toward outer space. The height attained by each rocket before all fuel was burned was recorded as a function of the weight of the fuel carried. Let the domain be the set of all weights of the fuel carried. Let the range be the heights attained. The following is the data which the scientists collected.

weight in pounds	height in miles
10,000	20
12,000	26
14,000	34
16,000	44
18,000	56

(a) Draw a horizontal and a vertical axis and place the appropriate labels for the domain and range on these axes.

(b) Plot the ordered pairs which the scientists collected when doing the experiment.

(c) Connect these points with a "smooth" curve.

(d) What height would you predict for another rocket carrying 15,000 pounds of fuel?

Answers to Sample Test Items

1. (a) (George Washington, 57)
(John Adams, 61)
(Thomas Jefferson, 57)
(William Taft, 51)
(Woodrow Wilson, 56)
(Franklin Roosevelt, 51)
(Harry Truman, 60)

(b) 7
(c) 7
(d) 5
(e) relation, function

2. $\{(3,1), (9,3), (12,4), (18,6), (\frac{3}{2},\frac{1}{2}), (1,\frac{1}{3}), (\frac{2}{3},\frac{1}{9})\}$

First element is three times the second element.

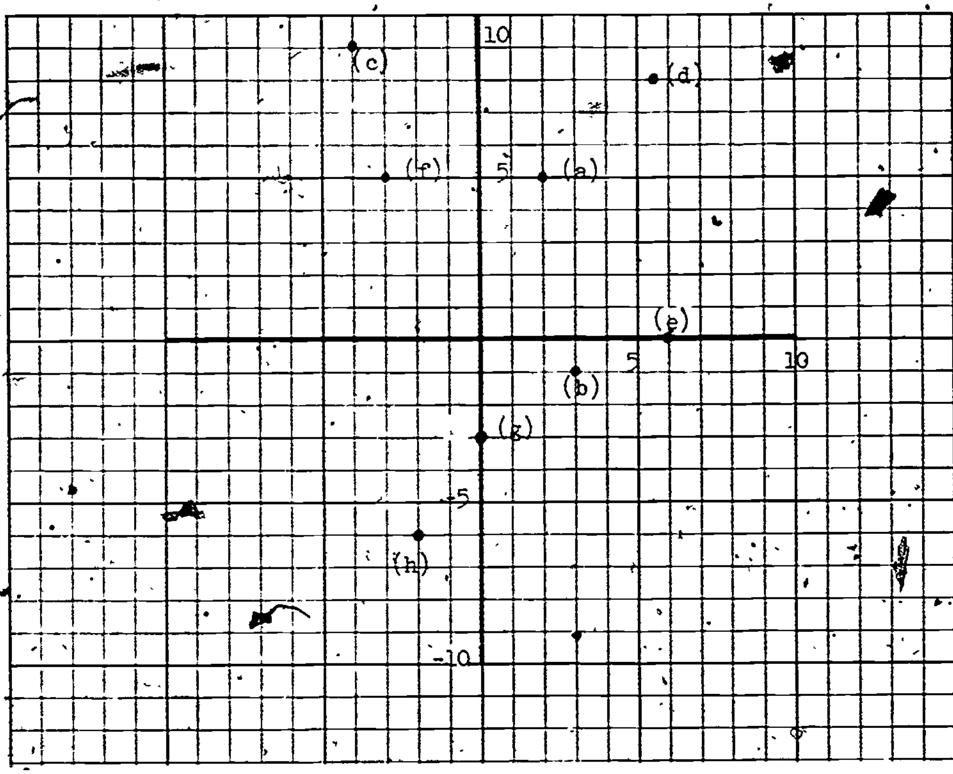
3. (a) $\{(0,1.1), (0, 1.4), (2,1.6), (3,1.9), (5,2)\}$

   (b) $\{(1,\frac{1}{2}), (1\frac{1}{2},2\frac{1}{2}), (2,1\frac{1}{2}), (3,4)\}$

4. a, b, d are functions.

5. a and c represent functions.

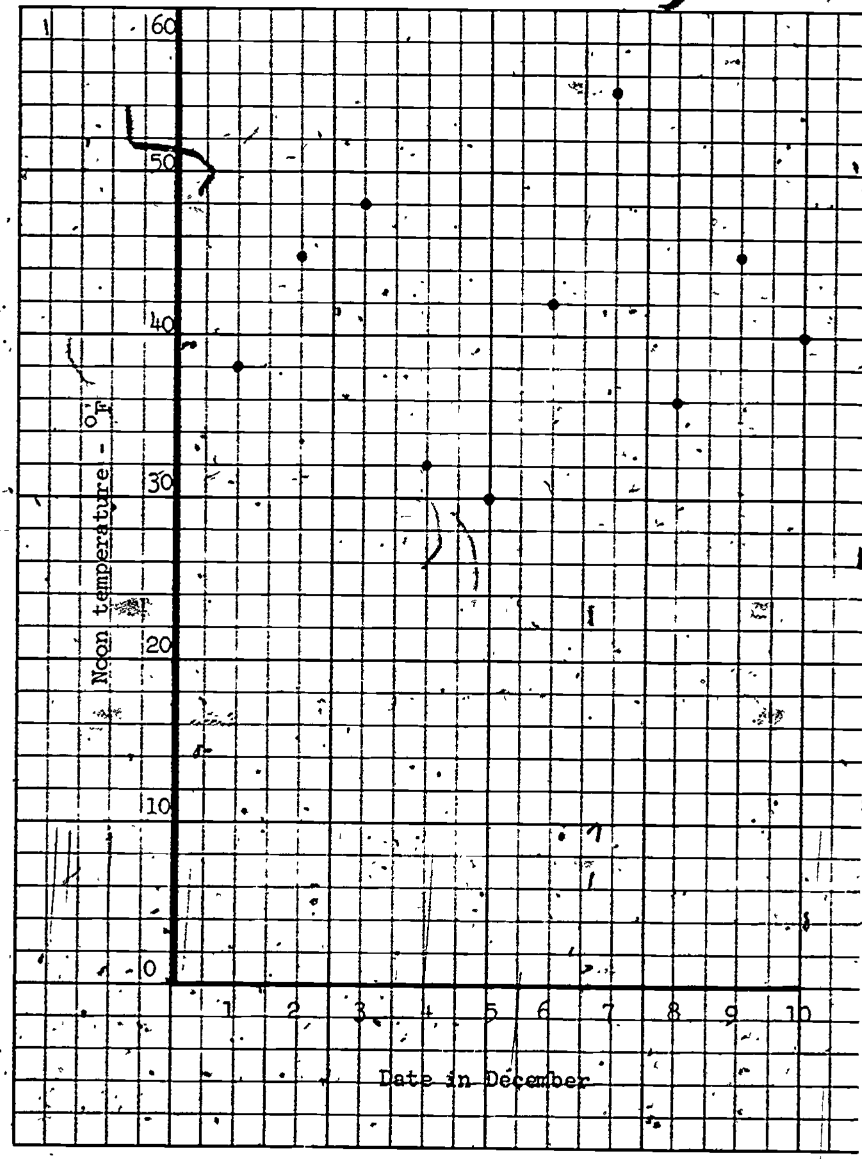
6.



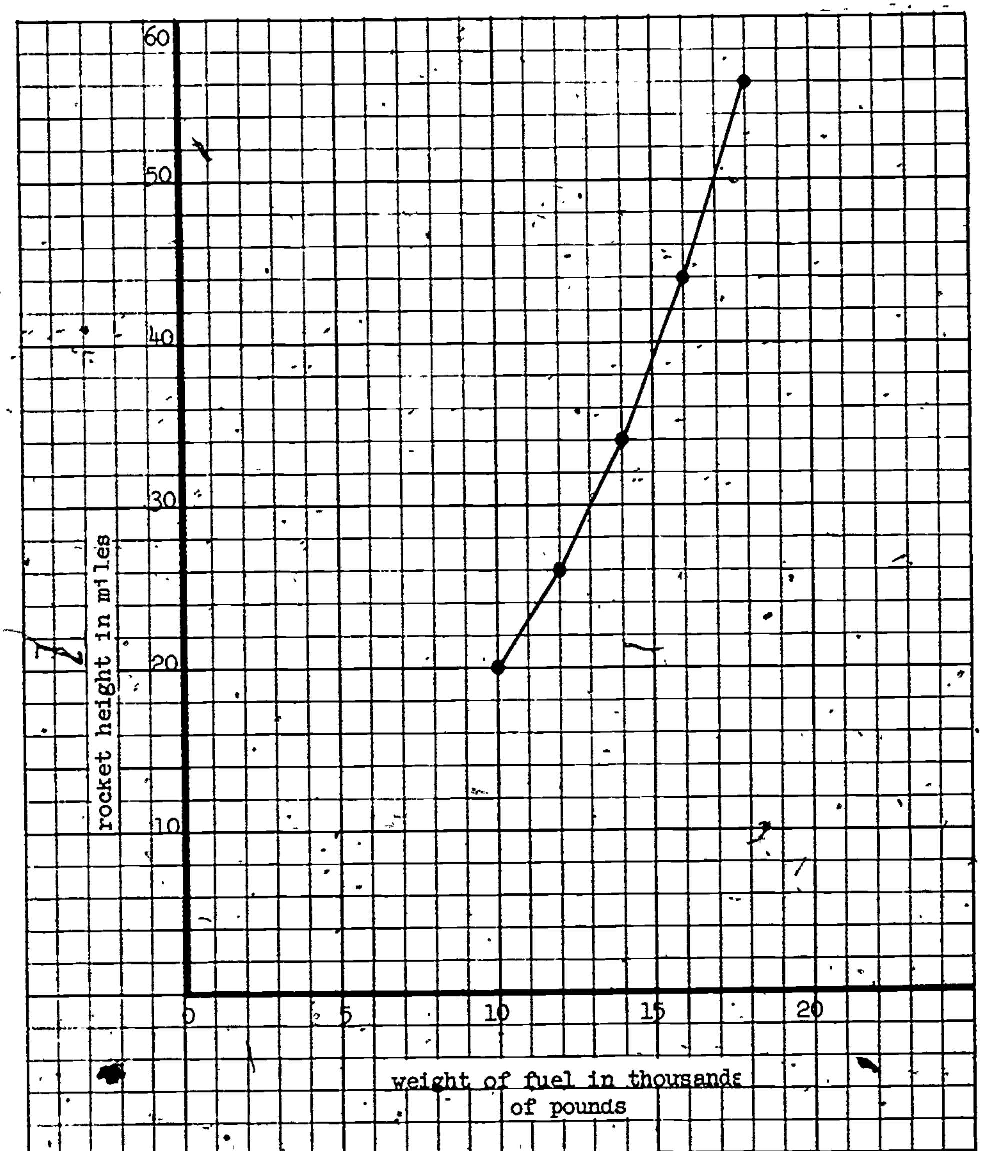
7. (a) first quadrant
   (b) third quadrant
   (c) fourth quadrant
   (d) third quadrant
   (e) second quadrant
   (f) second quadrant
   (g) first quadrant
   (h) negative x-axis
   (i) positive y-axis

8. (A) (5,3)
(B) (9,0)
(C) (1,11)
(D) (-8,6)
(E) (-10,2)
(F) (-3,0)
(G) (-6,-5)
(H) (0,-8)
(I) (3,-2)
(J) (9,-13)

9.



10.


60
50
40
30
20
10
rocket height in miles
0
5
10
15
20
weight of fuel in thousands of pounds

Chapter 4

THE LINEAR FUNCTION

In this chapter the major mathematical concepts which are presented are as follows:

(1) Graphing linear functions

(2) Representing linear functions as English sentences

(3) Expressing linear functions as equations of the form $y = mx$

(4) Slope of a linear function

(5) Linear functions which do not contain the ordered pair $(0,0)$, and expressing these functions as equations of the form $y = mx + b$.

4.1 Graphing Linear Functions Through the Origin

Students are led by examples like those cited in the first paragraphs of Section 4.1 to realize the existence of simple direct proportions. The student then performs the chalk-pencil experiment. The purpose of this experiment is to find out what the graph of a direct proportion looks like. He discovers it is a straight line which approaches the origin. The fact that it doesn't actually contain the origin may not be immediately obvious to the student. The length of the chalk expressed in miles is approximately 4.73×10^{-5} miles while the length of the pencil expressed in miles is approximately 10.9×10^{-5} miles. In scientific notation this would read 1.09×10^{-4}. This gives an ordered pair which for all practical purposes is $(0,0)$. Expressing these same measures in units such as light-years would yield a new ordered pair which would still not be $(0,0)$ but each element of this ordered pair would be much closer to zero than the element of the ordered pair was when the unit of measure was the mile. In a case such as this, note that the graph of the function does not contain the origin but this graph does approach the origin as a limit.

Description of chalk-pencil experiment.

(a) Each student does this experiment individually. (Whenever students work in groups each one should make his own data table and graph.)

(b) Equipment needed:

1 unused stick of chalk approximately 3" in length.

1 unsharpened pencil approximately 7" in length.

10 or more items with equally-spaced marks on them. A few not listed in Table 1 of the text are: screen wire, locket chain, grocery store receipt. (These can be passed around, as each student does not need ten of his own.)

(c) Remarks about the experiment:

Student should be instructed, on the day preceding the experiment, to look around his home for measuring scales. Any object or piece of paper that has equal spaces on it will serve.

To do the experiment the student measures the length of the chalk and pencil with each scale. Here, for example, is an illustration of how the chalk could be measured with a sheet of lined notebook paper.

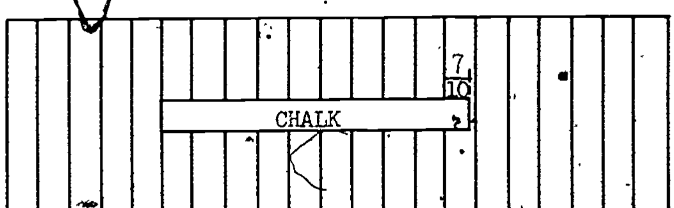


Figure 1

The student lays the chalk on the paper so that one end of the chalk is on a line. He counts the number of spaces completely spanned. In this case it is 9. He then estimates, to the nearest tenth, the fraction of the final space covered. It is $\frac{7}{10}$ in this example. The measure of the length of the chalk is $9\frac{7}{10}$ spaces. The student then measures the pencil's length with the same sheet of lined paper. Suppose the length of the pencil is $22\frac{3}{10}$ spaces. The ordered pair $(9\frac{7}{10}, 22\frac{3}{10})$ is recorded in the row labeled "notebook paper" in Table 1.

Measuring device	Length measure of the chalk	Length measure of the pencil
Ruler (inches)	3	7
Ruler (centimeters)	7.6	17.8
Ruler (feet)		
Ruler (meters)		
Notebook paper	$9\frac{7}{10}$	$22\frac{3}{10}$
Ordinary graph paper		
Thermometer		
Crossword puzzle		
Graduated cylinder		
Lines on a printed page		
Bead chain (bath-tub stopper)		

Table 1 (Text)

Note the ratio

$$\frac{\text{pencil length}}{\text{chalk length}} = \frac{22.3}{9.7} = 2.3$$

Scales with small subdivisions, like a thermometer, will give large measures. Scales with relatively large subdivisions like a ruler calibrated in inches, will give small measures. Every measuring scale will yield an ordered pair of chalk-pencil length measures whose ratio is approximately 2.3 . Hence, the graph of this function is a set of points that form a straight line which includes the origin and whose slope is 2.3 .

To help convince the student that the line approaches (0,0), have him measure the lengths in meters. The numbers in the pair will be very small, approximately (.08, .18). This point will be almost indistinguishable from zero on the student's graph.

The main purpose of this experiment is to reveal to the student that this kind of relation has a straight line graph that approaches the origin. No further mathematical analysis is done on the graph. However, the student should save his data table and graph because they will be referred to in future work.

The continuity of the chalk-pencil function is questioned. It is found to be continuous over the domain of all positive real numbers. The graph is a half line in the first quadrant. From the example given in this commentary, the slope of this half line is 2.3 . The origin is the point from which this half line starts but is not actually a point of the half line.

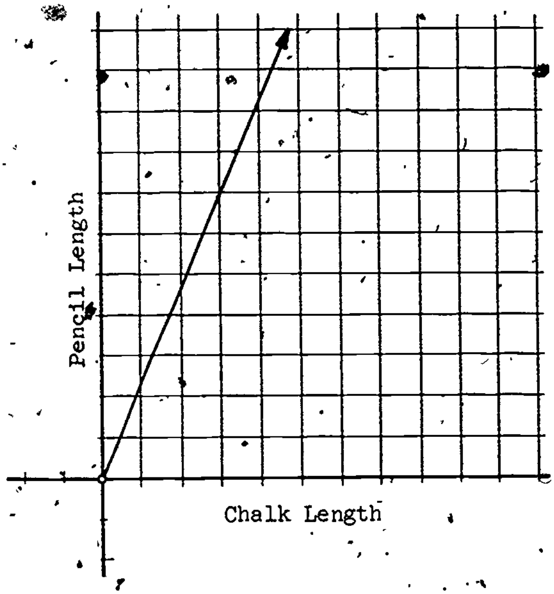


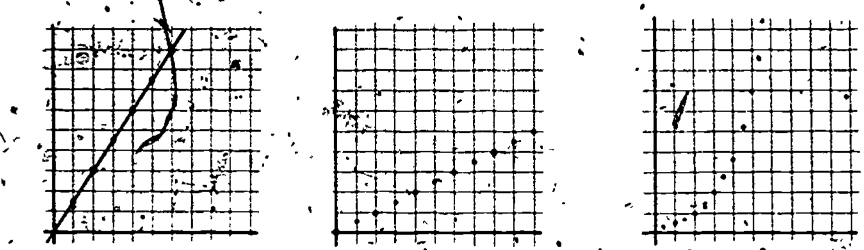
Figure 2

To emphasize the concept of continuity, an example (2 × 4's) of a discontinuous function is given. In this case the domain is a subset of the positive integers. Although the points of the discrete function described here are colinear, we will not classify this as a linear function. We will reserve this title for functions which are continuous over some interval of the real numbers and which are satisfied by the relation $y = mx + b$. The student also looks for other functions whose graphs are straight lines which approach the origin:

1. Inch and foot measures of various lengths.
2. Minute and second measures of various time intervals.

Exercise 1

1. Which of these graphs represents a linear relation?

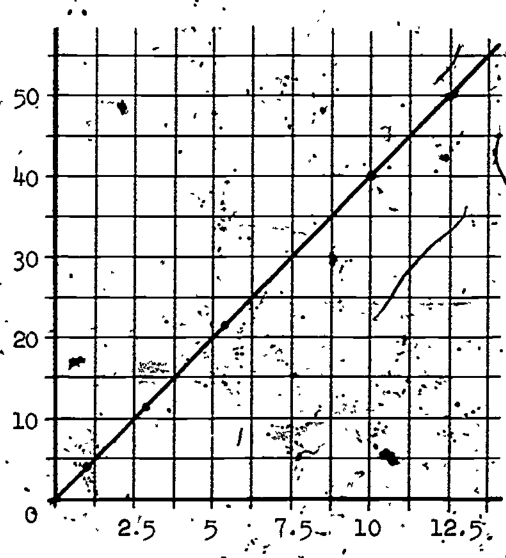


(a) Linear Function (b) Discrete Function (c) Discrete Function

2. (a) Supply the numbers missing from the table.

A	1	4
B	3	12
C	10	40
D	15	60
E	5.5	22
F	12.5	50

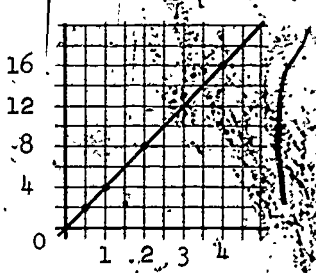
(b) Graph the data to see if the relation is linear.



3.

B	A
20	4
10	2
5	1
0	0
25	5

4. The following graph shows that the function is linear.



4.2 Representing Linear Functions by Sentences

In this section the student learns that any linear function whose graph includes the origin can be written as a sentence of the form:

$$\text{Value of one variable} = \text{Some Number} \times \text{Value of the other variable}$$

Exercise 2

1. Fill in the missing numbers:
   (a) Number of feet = 5,280 × Number of miles.
   There are 15,840 feet in 3 miles.
   (b) Number of quarts = 4 × Number of gallons.
   There are 32 quarts in 8 gallons.
   (c) Number of hours = 24 × Number of days.
   There are 6 hours in $\frac{1}{4}$ of a day.
   (d) Number of ounces = 16 × Number of pounds.
   There are 320 ounces in 20 pounds.
   (e) Diameter of a circle = 2 × radius.
2. Write the relation between:
   (a) Pounds measure and the corresponding tons measure.
   no. of pounds = 2000 × no. of tons
   (b) Foot measure and the corresponding yard measure.
   no. of feet = 3 × no. of yards
   (c) Hours measure and the corresponding minutes measure.
   no. of minutes = 60 × no. of hours
   (d) Cubic foot measure and the corresponding cubic yard measure.
   no. of cu. ft = 27 × no. cu. yards
   (e) Year measure and corresponding day measure.
   no. of days = 365 × no. of years
   (f) The circumference and the corresponding diameter of a circle.
   circumference = π × diameter
3. No. of feet = 3.28 × no. of meters.
4. No. of gallons = $7\frac{1}{2}$ × no. of cu. ft.

4.3 Functions of the Form: $y = mx$

The third step in the sequence of ideas on the straight line approaching (0,0) is to replace the words in the sentence representation by letters to get an equation of the form

$$y = mx .$$

Exercise 3

1. For the linear relation: $m = 60 h$
   (a) If $h = 3$ hours, $m =$ 180 minutes.
   (b) If $h = \frac{1}{2}$ hour, $m =$ 30 minutes.
   (c) If $m = 300$ minutes, $h =$ 5 hours.
   (d) If $m = 20$ minutes, $h =$ 1/3 hours.

2. A car averages 20 miles on a gallon of gas. An equation relating number of miles traveled to number of gallons of gas used is $m = 20 g$.

3. Complete the table using the linear relation: $y = 4x$.

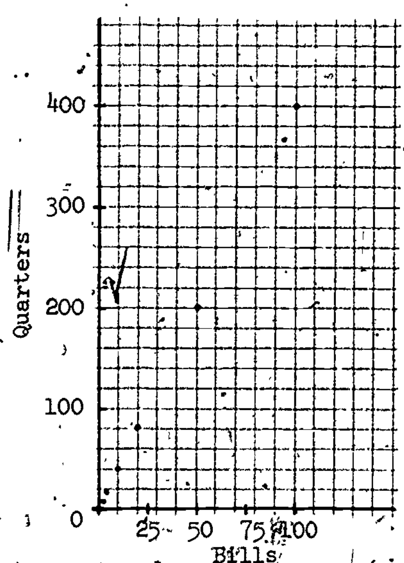
x	y
3	12
$\frac{1}{2}$	2
3	12
5.5	22
2.5	10

4. U.S. paper money is available in bills of the following denominations: \$ 1, \$ 2, \$ 5, \$ 10, \$ 20, \$ 50, and \$ 100.

   (a) Table showing equivalent number of quarters to each bill.

Bills	Quarters
\$1	4
\$2	8
\$5	20
\$10	40
\$20	80
\$50	200
\$100	400

   (b) Graph of table given in part (a).



(c) The graph is not continuous.

(d) An equation relating the value of any bill in dollars and the equivalent number of quarters is $q = 4d$.

4.4 Slope

We are now ready to study one of the important properties of a line which corresponds to the idea of the steepness of inclination of a line. The steepness of a stairway depends on the relationship between the rise and the run of a step.

AP = RISE
PB = RUN

Figure 3

If one stairway has steps with a certain rise and run and another stairway has steps with rise and run each twice as large, is it clear that the steepness of the two stairways is the same? In other words, a run of 2 with a rise of 1 gives the same steepness as a run of 4 with a rise of 2 .

Figure 4

The steepness or pitch of these stairways may be defined as the number obtained by dividing the rise by the run, $\frac{1}{2}$ in either case.

The concept of the slope of a line is based on the idea of a "rise divided by run". If we think of one step connecting two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ on a nonvertical line, then the rise is $|y_2 - y_1|$ and the run is $|x_2 - x_1|$.

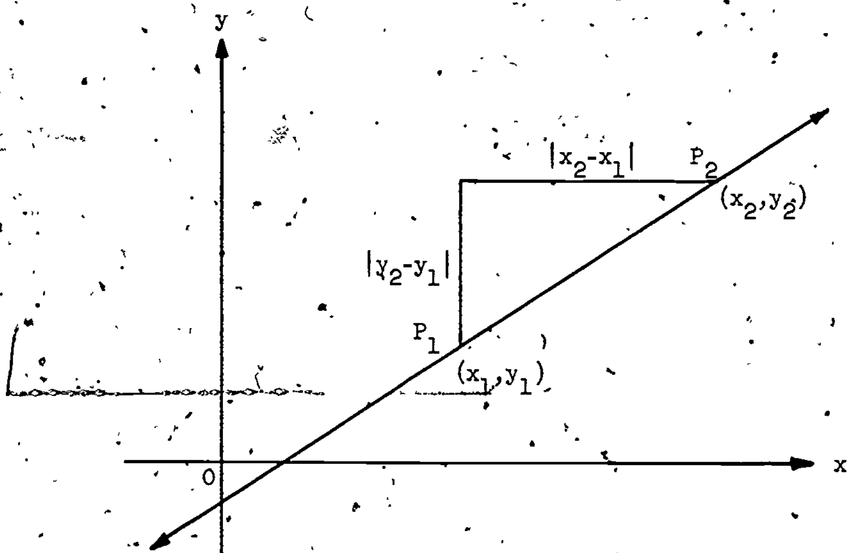


Figure 5

We could define the slope of the segment $\overline{P_1P_2}$ as rise divided by run, i.e., $\left|\frac{y_2 - y_1}{x_2 - x_1}\right|$. But we do not. The slope of a segment $\overline{P_1P_2}$ is defined as $\frac{y_2 - y_1}{x_2 - x_1}$. The formula without the absolute value is easier to handle, and it turns out to be more useful. The absolute value of the slope conveys only the magnitude of the slope. The sign of the slope conveys the additional idea of "slopes up or down" as suggested in Figure 6.

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The concept of the slope of a line is based on the idea of a "rise divided by run". If we think of one step connecting two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ on a nonvertical line, then the rise is $|y_2 - y_1|$ and the run is $|x_2 - x_1|$.

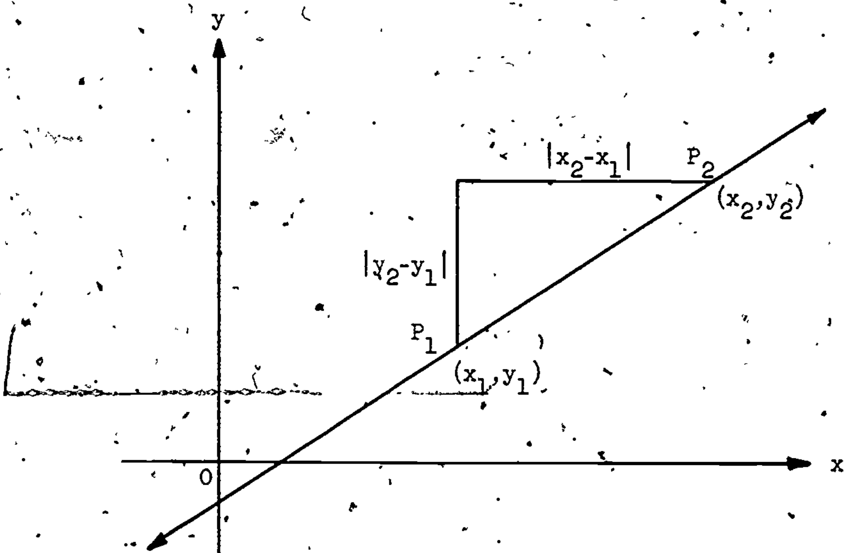

Figure 5

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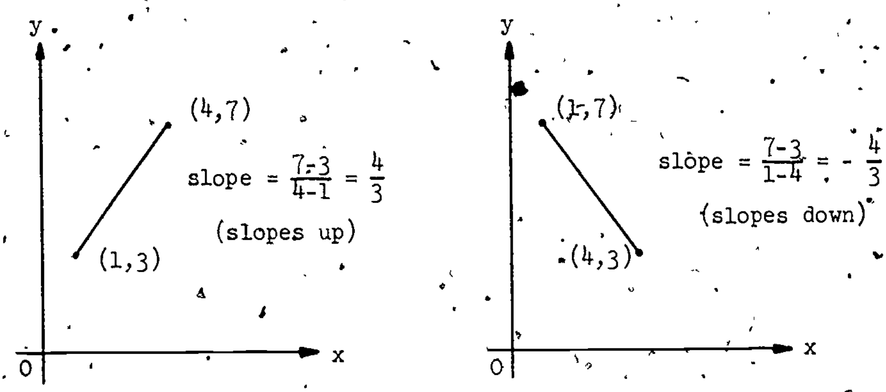


Figure 6

Starting with the concept of the slope of a segment, we now develop the concept of the slope of a line. Consider the line $\overleftrightarrow{AB}$ where A = (1, 2) and B = (3, 5). Then $\overleftrightarrow{AB} = \{(x, y) : y = \frac{3}{2}x + \frac{1}{2}$ and x is real$\}$. Note that the slope of $\overleftrightarrow{AB}$ is $\frac{5-2}{3-1} = \frac{3}{2}$. If we consider the coordinate of other points of $\overleftrightarrow{AB}$ such as $(a, \frac{3a+1}{2})$ and $(b, \frac{3b+1}{2})$ the slope of the segment joining this point is $\dfrac{\frac{3b+1}{2} \quad \frac{3b+1}{2}}{b-a} = \frac{3}{2}$.

Every nonvertical line has the property that all of its segments have the same slope. Let c be any line such that $\overleftrightarrow{c} = \{(x,y) : y = mx + b$ for $m \neq 0\}$. Then a point R of this line has the coordinates (r, rm + b) and a point S has the coordinates (s, sm + b). The slope of $\overline{RS}$ is

$\frac{(sm + b) - (rm + b)}{s - r} = \frac{sm - rm}{s - r} = m\left(\frac{s-r}{s-r}\right) = m$. This proves that all segments of a nonvertical line have the same slope. We may then write the following definition and theorem:

<u>DEFINITION</u>: The slope of a nonvertical line is equal to the slope of any of its segments.

<u>THEOREM</u>: The slope of a nonvertical line p is $\frac{y_2 - y_1}{x_2 - x_1}$, where $\overline{P_1P_2}$ is any segment of p and $P_1 = (x_1, y_1)$, $P_2 = (x_2, y_2)$.

We now consider three possibilities for the slope of a line; it is positive, it is zero, or it is negative. Let $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ be two points of a line. We suppose that the points are named so that P_2 has

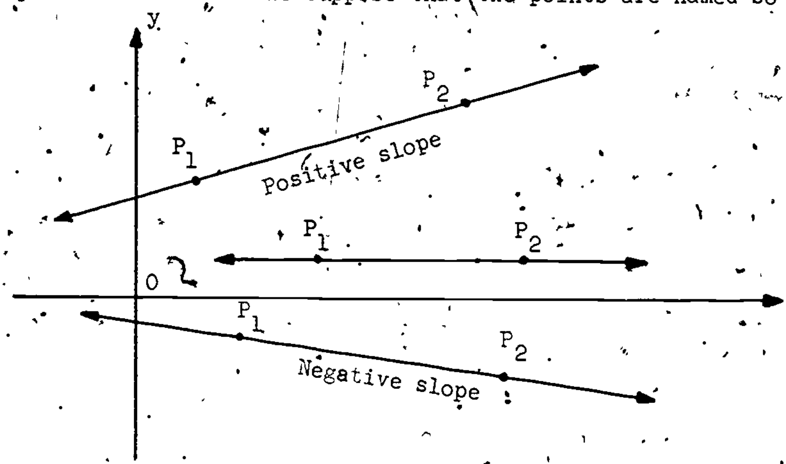


Figure 7

the larger x-coordinate. We disregard the possibility $x_2 = x_1$, since this would imply that $\overleftrightarrow{P_1P_2}$ is a vertical line and the slope of a vertical line is undefined. That is, we say a vertical line has no slope.

Possibility 1. The slope is positive. Since we named points so that $x_2 > x_1$ and $y_2 > y_1$, it follows that $x_2 - x_1$ and $y_2 - y_1$ are both positive. This means that as a particle moves along $\overleftrightarrow{P_1P_2}$ from left to right (from the point with x-coordinate x_1 to the point with x-coordinate x_2), it is going uphill.

If $x_2 < x_1$ and $y_2 < y_1$ then $x_2 - x_1$ and $y_2 - y_1$ are both negative. This means that P_1 is to the right and up from P_2 and the situation is unchanged. Hence, the slope is also positive.

Possibility 2. The slope is zero. Then $y_2 - y_1 = 0$. This means intuitively that, as a particle moves along the ~~line~~ $\overleftrightarrow{P_1P_2}$, it is moving on "level ground". (The y-coordinates of all the points of the line are the same.)

Possibility 3. The slope is negative. Then one of the numbers, $y_2 - y_1$ and $x_2 - x_1$, is positive and the other one is negative. Since we named the points so that $x_2 > x_1$ it follows that $x_2 - x_1$ is positive and $y_2 - y_1$ is negative, that is, $y_2 < y_1$. This means intuitively that, as a particle moves along $\overleftrightarrow{P_1P_2}$ from left to right, it is going downhill.

Exercise 4

1. $m = \frac{16 - 12}{4 - 3} = 4$.

2. Line A, $m = \frac{10 - 5}{20 - 10} = \frac{5}{10} = \frac{1}{2}$.

   Line B, $m = \frac{15 - 10}{15 - 10} = \frac{5}{5} = 1$.

   Line C, $m = \frac{20 - 10}{10 - 5} = \frac{10}{5} = 2$.

   Line D, $m = \frac{25 - 10}{5 - 2} = \frac{15}{3} = 5$.

3. Using $(1\frac{1}{2}, 90)$ and $(2, 120)$,

   $m = \frac{120 - 90}{2 - 1\frac{1}{2}} = \frac{30}{\frac{1}{2}} \cdot \frac{2}{2} = 60$.

4. If you were to graph the following equations, what would be the slopes of the lines?

   (a) $d = 365y$

   $m = 365$

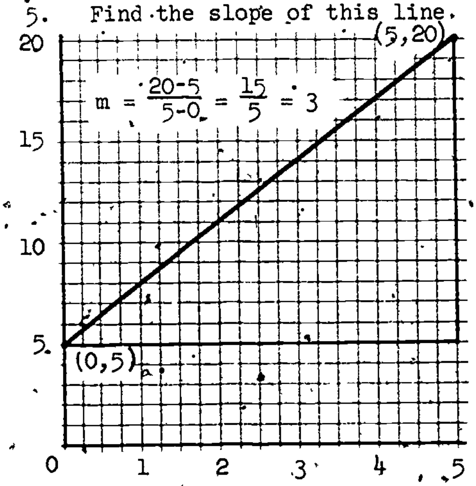
   (b) $o = 16p$

   $m = 16$

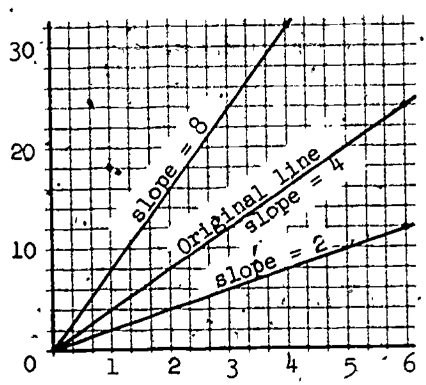
   (c) $y = 3x$

   $m = 3$

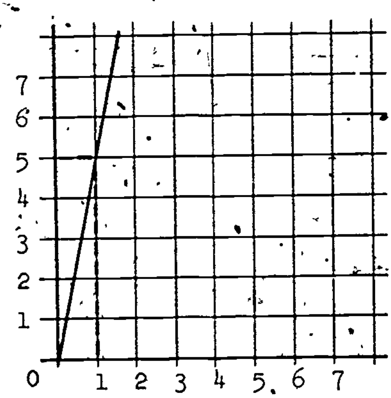
5. Find the slope of this line.



6. Draw a line that has twice the slope of the line shown on the graph. Draw another line with half the slope.



7. Graph the equation $y = 5x$ using the fact that the slope is 5.



4.5 Coat Hanger Experiment

Students should do this experiment in groups of four.

The equipment needed for each group of four is as follows:

1 pencil
1 strip of Scotch or masking tape
1 coat hanger
2 jumbo size (2" long) paper clips
1 centimeter ruler
1 set of weights

A critical part of the apparatus is the hook made from two paper clips. Two clips are needed so that opposing pointers counter-balance each other and give a true "zero" reading when there is no object hanging on them.

The total load of 1000 gm causes the hanger to bend a total of between 20 and 30 mm. The bend is proportional to the load up to about 1400 gm. For loads greater than this the graph departs from its initial linearity. Since this section is devoted to linear relations whose graphs include the origin, the coat hanger load is restricted to a maximum of 1000 gm.

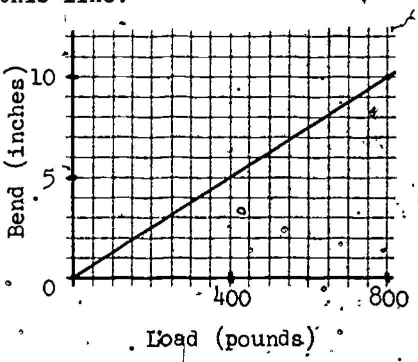
Exercise 5

1. Suppose there is a linear relation between the amount a diving board bends and the load on it. Then it would be like the coat hanger.
   (a) If the board bends down 1.5 inches when you (120 pounds) get on it, what would be the total bend when your friend, who also weighs 120 pounds, joins you? <u>3 inches</u>
   (b) Suppose you exert a force of 400 pounds on the board when you jump on it. How much will it bend? <u>5 inches</u>

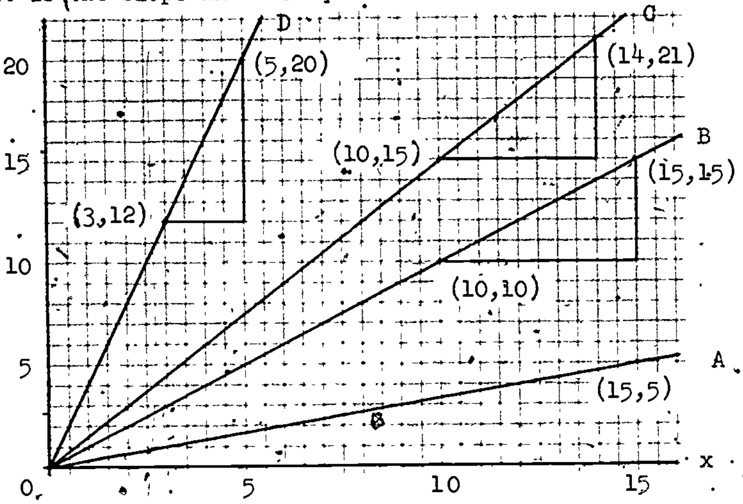
2. Make a data table and a graph of the linear function described in Problem 1. What is the slope of this line?

Load	Bend
120	1.5
240	3
400	5
800	10

$$m = \frac{5-3}{400-240} = \frac{2}{160} = \frac{1}{80}$$

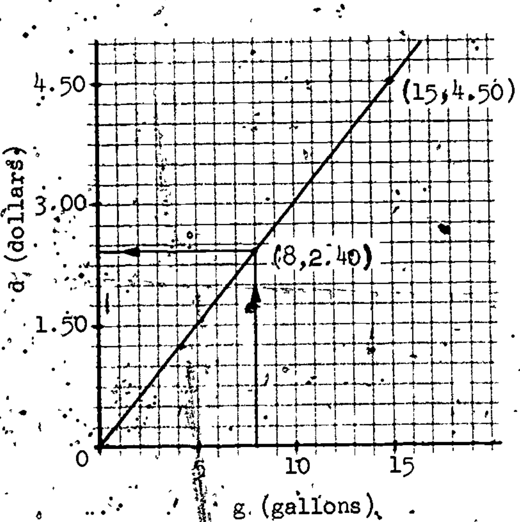


3. What is the slope and the equation of each of these lines?



	Slope	Equation
Line A:	$m = \frac{5-0}{15-0} = \frac{1}{3}$	$y = \frac{1}{3}x$
Line B:	$m = \frac{15-10}{15-10} = \frac{5}{5} = 1$	$y = x$
Line C:	$m = \frac{21-15}{14-10} = \frac{6}{4} = \frac{3}{2}$	$y = \frac{3}{2}x$
Line D:	$m = \frac{20-12}{5-3} = \frac{8}{2} = 4$	$y = 4x$

4. A gas station attendant could use the following graph to figure out how much to charge a customer for the gas he puts in his tank.



(a) What is the slope of the line?

$m = \frac{4.5}{15} = \frac{45}{150} = \frac{3}{10}$

(b) What is the cost of 8 gallons of gas?

\$ 2.40

(c) What is the cost of 6.5 gallons?

\$ 1.95

(d) Write an equation that could be used to figure gasoline bills.

$c = .30\,g$

4.6 Graphing Linear Functions in General - Spring Experiment

1 hook weight (100 gram)	1 pencil (unsharpened)
2 hook weights (200 gram)	1 roll Scotch tape
1 hook weight (500 gram)	1 foot-ruler, with metri scale
1 spring	2 sheets of note paper

This experiment investigates the relation between the length of a spring and the load hanging on it. The relation is linear for a good spring as long as its maximum length does not exceed three times its unstretched length. Although it is a straight line the graph of the relation will not pass through (0,0) because the spring's length is not zero when the load on it is zero.

Exercise 6

1. In the figure to the right, the four lines each have the same slope. The line through the origin has the equation

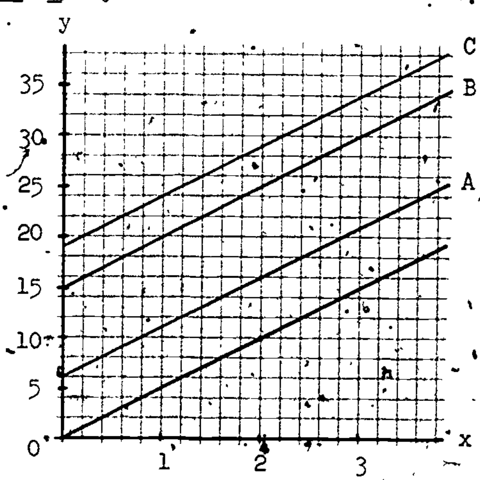
$$y = 5x .$$

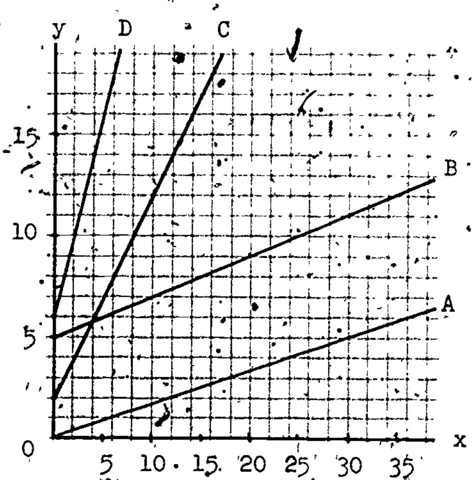
Write the equation of each of the other three lines.

Line A: $y = 5x + 6$

Line B: $y = 5x + 15$

Line C: $y = 5x + 19$



2. 

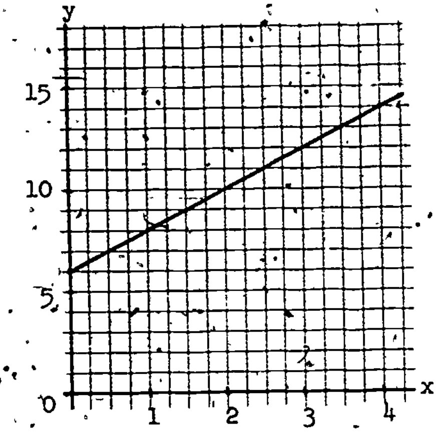
Use the accompanying figure to fill in this chart.

Line	m	b	y = mx + b
A	$\frac{1}{6}$	0	$y = \frac{1}{6}x$
B	$\frac{1}{6}$	5	$y = \frac{1}{6}x + 5$
C	1	2	$y = x + 2$
D	2	6	$y = 2x + 6$

3. The equation of a line is

$y = 2x + 6$.

Graph this line.



4. $(1, 7)$, $(3, 19)$, $(2, 10)$ and $(\frac{1}{3}, 3)$ are on the line $y = 6x + 1$. $(\frac{1}{2}, 4)$ is not a point of this line.

4.7 The Centigrade-Fahrenheit Experiment

1 thermometer, Centigrade (-20° to 110°)

1 thermometer, Fahrenheit (0° to 230°)

1 No. 10 tin can (for liquids)

ice

salt

The experiment is begun by making the six water baths of varying temperatures mentioned in the text. Use at least half a can of water for each bath so that its temperature will not change excessively during the course of the experiment. (Heat loss or gain during the experiment introduces no error since any pair of C-F readings is taken simultaneously. In fact, temperature changes have the advantage of giving different students different data, yet they all get a straight line graph.)

The lowest temperature that can be obtained by putting salt with crushed ice is about -12 degrees Centigrade. The amount of salt needed to get down to this temperature depends on the volume of the mixture. Twenty heaping tablespoons should be sufficient for half a can of crushed ice. More salt than needed will have no adverse effect.

More than six water baths can be prepared if the teacher wants additional data. The temperature of the air in the classroom provides an ordered pair that can be entered in the table. Be sure the thermometers are dry; otherwise evaporation will probably cool them unequally.

Two procedures for getting the temperature readings suggest themselves:

(a) Students, in groups of four, can move from container to container, carrying their thermometers with them.

(b) The two thermometers can be left in each container and the students can form as many queues as there are containers. Each student reads each pair of thermometers. After any given pair of measurements, he moves to the rear of one of the other queues.

Exercise 7

1. Slope of the C vs F graph is

$$m = \frac{212 - 32}{100 - 0} = \frac{180}{100} = \frac{9}{5}.$$

2. The y-intercept of the C vs F graph is 32. Note (0, 32).

3. Using the results of Problems 1 and 2, equation $F = mC + b$ becomes $F = \frac{9}{5}C + 32$.

4. Use this equation to find F temperature that corresponds to given C temperature.

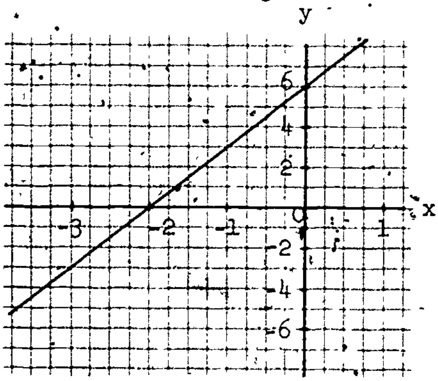
C	F
20	68
45	113
-5	23
0	32

5. Plot the points found in Problem 4 on the experimental graph. Do they fall on the C vs F line?

They certainly should.

6. (a) Plot the data in the table to the right.

x	y
-1	3
-2	0
-4	-6



(b) Find the slope and y-intercept of the line.

$m = 3$

$b = 6$

(c) Find the equation of this line.

$y = 3x + 6$

Sample Test Items

Part I: Place the word, phrase or number in each blank which will make the statement true.

1. Number of yards = ________ × Number of miles.
2. There are ________ yards in 2 miles.
3. \$ 2 = ________ nickels.
4. The equation, $y = 10x$, defines a ________ function.
5. In the graph of the equation, $y = 5x + 3$, the number 5 tells the ________ of the graph.
6. The slope-intercept form of the linear equation is ________.
7. The equation, ________, graphs as a straight line which passes through the origin.
8. The ordered pair (5, _____) is a point of the line whose equation is $y = \frac{1}{5}x - \frac{1}{5}$.
9. The line containing the points (5, 0) and (7, 2) has a slope of ________.
10. If $F = \frac{9}{5}C + 32$, then room temperature on the Fahrenheit scale is ________ when the Centigrade reading is 25.

Part II: Problems.

1. Complete the table using the linear relation $y = 3x$.

x	y
3	
$\frac{1}{3}$	
	12
0	
2	
	15

2. The coordinates of 4 points are given A(3, 9), B(4, 12), C(5, 16), D(7, 21). Find the slope of each of the following lines.

Line AB m = __________

Line AC m = __________

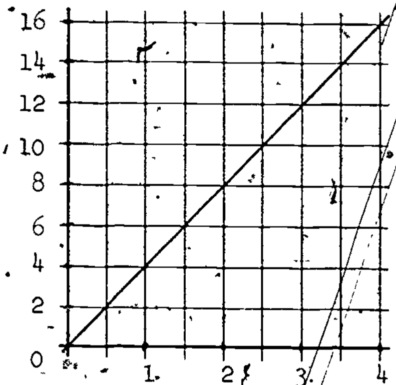
Line AD m = __________

Which set of 3 points all lie on the same straight line?

____

3. From the following graph, compute the slope and intercept, and write the equation for the linear function.

(a)

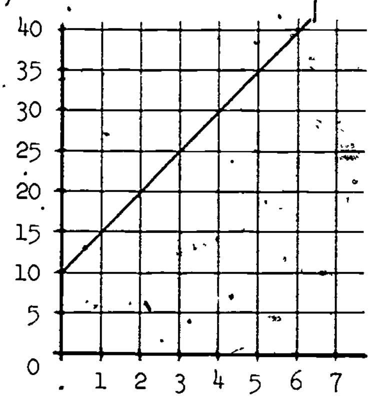


m = __________

b = __________

Equation: __________

(b)



m = __________

b = __________

Equation: __________

4. Graph these points on graph paper and connect them with a straight line. Find the slope, the intercept and the equation of the line.

x	y
0	6
1	8
2	10
3	12
4	14

m = __________

b = __________

Equation: __________

5. Fill in the following table for each of the following sequences of ordered pairs.

	Functions yes/no	Linear yes/no	If linear slope
(a) (0, 1), (1, 2), (2, 3), (3, 4), (4, 5)			
(b) (0, 0), (1, 1), (2, 4), (3, 9), (4, 16)			
(c) (0, 0), (3, 1), (4, 2), (5, 5), (4, 8)			
(d) (0, 2), (2, 5), (6, 11), (8, 14), (4, 8)			
(e) (0, 4), (9, 4), (10, 4), (12, 4), (13, 4)			
(f) (3, 1), (3, 2), (3, 3), (3, 4), (3, 5)			

Solutions for Sample Test Items

Part I

1. 1760
2. 3520
3. 40
4. linear
5. slope
6. y = mx + b
7. y = mx
8. $\frac{4}{5}$
9. 1
10. 77

Part II

1.

x	y
3	9
$\frac{1}{3}$	1
4	12
0	0
2	6
5	15

2.

Line AB $m = \frac{12 - 9}{4 - 3} = 3$

Line AC $m = \frac{16 - 9}{5 - 3} = \frac{7}{2}$

Line AD $m = \frac{21 - 9}{7 - 3} = \frac{12}{4} = 3$

Points A, B and D all lie on the same straight line.

3. (a) $m = \frac{16 - 8}{4 - 2} = \frac{8}{2} = 4$

$b = 0$

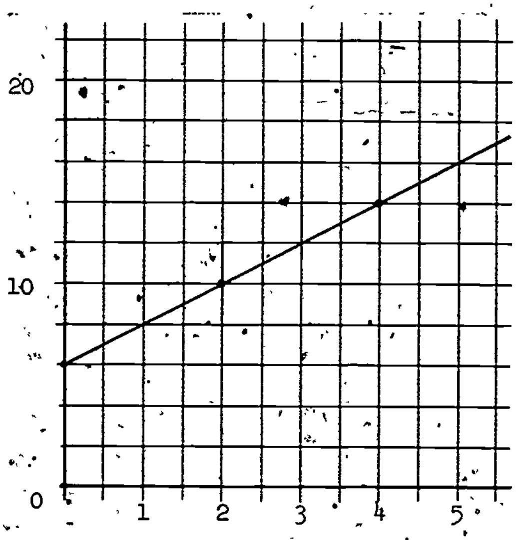
Equation: $y = 4x$

(b) $m = \frac{40 - 20}{6 - 2} = \frac{20}{4} = 5$

$b = 10$

Equation: $y = 5x + 10$

4.



$m = \frac{14 - 10}{4 - 2} = \frac{4}{2} = 2$

$b = 6$

Equation: $y = 2x + 6$

5.

	Function yes/no	Linear yes/no	If linear slope
(a)	yes	yes	1
(b)	yes	no	
(c)	no	no	
(d)	yes	yes	$\frac{3}{2}$
(e)	yes	yes	horizontal line
(f)	no	yes	vertical line